Numerical Solution of Linear System of Fredholm Integral Equations Using Haar Wavelet Method

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Abstract:

The aim of this paper is to present the numerical method for solving linear system of Fredholm integral equations, based on the Haar wavelet approach. Many test problems, for which the exact solution is known, are considered. Compare the results of suggested method with the results of another method (Trapezoidal method). Algorithm and program is written by Matlab vergion 7.

Keywords: linear system of Fredholm integral equations, Haar wavelet method

1. Introduction:

Integral equations are used as mathematical models for many physical situations, and integral equations also occur as reformulations of other mathematical problems. Recently a great deal of interest has been focused on the solution of integral equations by the wavelet methods, the first paper which used Haar wavelet method to solve integral equation has presented in 1991 by Beylkin, G. [1]. After that many researches are presenting this method to solve other types of integral equations. In [2] Haar wavelet method is applied to solve different types of linear integral equations (Fredholm, Volterra, integro-differential, weakly singular integral equations), also the eigenvalue problem is solved, while [3] applied the Haar wavelet method to solve the nonlinear Fredholm integral equation, and [4] used the wavelet for solving Fredholm integral equation. [5] presented the application of the Haar wavelet transform to solving integral and differential equations, while [6] used the rationalized Haar functions method for solving linear of Fredholm integral equations system.

In this paper the Haar wavelet method is applied to solving linear system of Fredholm integral equations. The method is first applied to an equivalent integral equations system, where the solution is approximated by Haar wavelet function with unknown coefficients. The collocation method is used to evaluate the unknown coefficients and find the approximate solution.

The method is tested with the aid of three numerical examples, for which the exact solution is known.

2. Haar wavelet method:

Haar functions have been used from 1910 when they were introduced by the Hungarian mathematician Alfred Haar. Haar wavelets are the simplest wavelets among various types of wavelets. They are step functions (piecewise constant functions). Haar wavelet form an orthogonal and complete set of functions representing discredited functions and piecewise constant functions. [7]

The set of Haar functions is defined as a group of square waves with magnitude ± 1 some intervals and zero elsewhere

$$h_i(t)=\begin{cases}1 & for \quad t\in[t^{(1)},t^{(2)}) \\ -1 & for \quad t\in[t^{(2)},t^{(3)}) \\ 0 & elsewhere\end{cases} \qquad \text{……………(1)}$$

Here the notations [2]

$$t^{(1)}=\frac{k}{m}\ ,\ t^{(2)}=\frac{k+0.5}{m},\ t^{(3)}=\frac{k+1}{m} \qquad \text{…………….(2)}$$

are introduced. The integer $m=2^j$, $j=0,1,...,J$, indicates the level of wavelet; $k=0,1,2,...,m-1$ is th translation parameter. The integer J determines the maximal level of resolution. The index i is calculated from the formula $i=m+k+1$. Here the minimal value is $i=2$ (then $m=1, k=0$); the maximal value is $i=2M$, where $M=2^J$. the index $i=1$ corresponds to the scaling function

$$h_1(t)=\begin{cases}1 & for \quad 0\le t<1 \\ 0 & elsewhere\end{cases} \qquad \text{…………….(3)}$$

Let us divide the interval $t\in[0,1]$ into $2M$ parts of equal length $\Delta t=1/(2M)$ and introduce the collocation points

$$t_l=(l-0.5)/(2M), \qquad l=1,2,...,2M \qquad \text{…………….(4)}$$

Following Chen and Hsiao [8], the Haar coefficient matrix H is introduced. It is a $2M\times 2M$ matrix with the elements $H(i,l)=h_i(t_l)$. A function $u(x)$ which is defined in the interval $t\in[0,1]$ can be expanded into the Haar wavelet series:

$$u(x)=\sum_{i=1}^{2M}a_ih_i(x) \qquad \text{…………….(5)}$$

where a_i are the wavelet coefficients. The discrete from of this equation is

$$u(x_l)=\sum_{i=1}^{2M}a_ih_i(x_l)=\sum_{i=1}^{2M}a_iH_{il}$$

or in a matrix presentation $u=aH$ where u and a are 2M dimensional row vectors.

3. Fredholm integral equations: [2]

A linear Fredholm integral equation has the form

$$u(x)-\int_0^1 K(x,t)u(t)dt=f(x), \qquad x\in[0,1] \qquad \text{…………….(6)}$$

Where f(x)are given function of x, k(x, t)is a function of two variables x and t called kernel of integral equation which are also known, while u(x) is to be determined.

Replacing (5) into (6) we find:

$$\sum_{i=1}^{2M} a_i h_i(x) - \sum_{i=1}^{2M} a_i G_i(x) = f(x) \qquad \text{…………….(7)}$$

where

$$G_i(x) = \int_0^1 K(x,t) h_i(t)dt \qquad \text{…………….(8)}$$

Next we shall evaluate the wavelet coefficients a_i by the collocation method. Satisfying (7) only at the collocation points (4) we get a system of linear equations

$$\sum_{i=1}^{2M} a_i \left[h_i(x_l) - G_i(x_l)\right] = f(x_l) \ , \qquad l = 1,2,...,2M \qquad \text{…………….(9)}$$

the matrix from of this system is

$$a(H-G) = F \qquad \text{…………….(10)}$$

where $G_{il} = G_i(x_l)$, $F_l = f(x_l)$

Example (1): consider the equation

$u(x) - \int_0^1 (x+t)u(t)dt = x^2$ where the exact solution is $u(x) = x^2 - 5x - 17/6$

Taking into account (1) and evaluating the integrals (8) we find

$$G_i(x) = \begin{cases} x + 0.5 & for \quad i = 1 \\ -\dfrac{1}{4m^2} & for \quad i > 1 \end{cases}$$

If we apply the collocation method, then the vector a can be calculated from the system (10).

Computations were carried out for different values of J. The accuracy of the results was estimated by the least square error, the errors are shown in Table (1).

J	2M	L.S.E.
2	8	0.0045
3	16	2.8743e-004
4	32	1.8246e-005

Table (1)

4. Linear system of Fredholm integral equations:

Consider the following linear Fredholm integral equation system:

$$u_r(x) - \sum_{s=1}^{n} \int_0^1 k_{rs}(x,t) u_s(t)dt = f_r(x) \qquad r = 1,2,...,n \qquad \text{…………….(11)}$$

where $f_r \in L^2[0,1]$, $k_{rs} \in L^2([0,1]\times[0,1])$ for $r,s=1,2,...,n$ and u_r are unknown functions [9].

we can be written it as the form

$$U(x)-\int_0^1 K(x,t)U(t)dt = F(x) \qquad \text{…………….(12)}$$

where $U(x)=\begin{bmatrix} u_1(x) \\ u_2(x) \\ \vdots \\ u_n(x) \end{bmatrix}$, $F(x)=\begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix}$, $K(x,t)=\begin{bmatrix} k_{11}(x,t) & k_{12}(x,t) & \dots & k_{1n}(x,t) \\ k_{21}(x,t) & k_{22}(x,t) & \dots & k_{2n}(x,t) \\ \vdots & \vdots & \cdots & \vdots \\ k_{n1}(x,t) & k_{n2}(x,t) & \dots & k_{nn}(x,t) \end{bmatrix}$

Replacing (5) into (11) we find

$$\begin{bmatrix} \sum_{i=1}^{2M} a_{1_i} h_i(x) \\ \sum_{i=1}^{2M} a_{2_i} h_i(x) \\ \vdots \\ \sum_{i=1}^{2M} a_{n_i} h_i(x) \end{bmatrix} - \begin{bmatrix} \sum_{i=1}^{2M} a_{1_i} \\ \sum_{i=1}^{2M} a_{2_i} \\ \vdots \\ \sum_{i=1}^{2M} a_{n_i} \end{bmatrix}^T \begin{bmatrix} \int_0^1 k_{11}(x,t)h_i(t)dt & \cdots & \int_0^1 k_{1n}(x,t)h_i(t)dt \\ \int_0^1 k_{21}(x,t)h_i(t)dt & \cdots & \int_0^1 k_{2n}(x,t)h_i(t)dt \\ \vdots & \ddots & \vdots \\ \int_0^1 k_{n1}(x,t)h_i(t)dt & \cdots & \int_0^1 k_{nn}(x,t)h_i(t)dt \end{bmatrix} = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix}$$

we can be written it as the form

$$\sum_{i=1}^{2M} a_{r_i} h_i(x) - \sum_{i=1}^{2M} a_{r_i} G_i(x) = F(x) \qquad \text{…………….(13)}$$

where

$$G_i(x)=\begin{bmatrix} G_{11}=\int_0^1 k_{11}(x,t)h_i(t)dt & \cdots & G_{1n}=\int_0^1 k_{1n}(x,t)h_i(t)dt \\ G_{21}=\int_0^1 k_{21}(x,t)h_i(t)dt & \cdots & G_{2n}=\int_0^1 k_{2n}(x,t)h_i(t)dt \\ \vdots & \ddots & \vdots \\ G_{n1}=\int_0^1 k_{n1}(x,t)h_i(t)dt & \cdots & G_{nn}=\int_0^1 k_{nn}(x,t)h_i(t)dt \end{bmatrix} \qquad \text{…….(14)}$$

next we shall evaluate the wavelet coefficients a_{r_i} by collocation method.

Satisfying (13) only at the collocation points (4) we get a system of linear equations:

$$\sum_{i=1}^{2M} a_{r_i} h_i(x_l) - \sum_{i=1}^{2M} a_{r_i} G_i(x_l) = F(x_l) \quad ,l=1,2,...,2M \ ,r=1,2,...,n \ \text{…….(15)}$$

The matrix form of this system is

$$\begin{bmatrix}\sum_{i=1}^{2M} a_{1_i}\\ \sum_{i=1}^{2M} a_{2_i}\\ \vdots \\ \sum_{i=1}^{2M} a_{n_i}\end{bmatrix}^T \left(\begin{bmatrix} H-G_{11} & -G_{12} & \cdots & -G_{1n}\\ -G_{21} & H-G_{22} & \cdots & -G_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ -G_{n1} & -G_{n2} & \cdots & H-G_{nn}\end{bmatrix}^T\right) = \begin{bmatrix} f_1(x)\\ f_2(x)\\ \vdots \\ f_n(x)\end{bmatrix}^T \qquad \ldots\ldots\ldots\ldots(16)$$

or

$$A(B)=F$$

where

$$A=\left[\sum_{i=1}^{2M} a_{1_i}\ \ \sum_{i=1}^{2M} a_{2_i}\ \cdots \sum_{i=1}^{2M} a_{n_i}\right],\quad B=\begin{bmatrix} H-G_{11} & -G_{21} & \cdots & -G_{n1}\\ -G_{12} & H-G_{22} & \cdots & -G_{n2}\\ \vdots & \vdots & \ddots & \vdots\\ -G_{1n} & -G_{2n} & \cdots & H-G_{nn}\end{bmatrix},\quad H(i,l)=h_i(t_l)$$

By solving linear system (16) we can find the wavelet coefficient a_{r_i} .

The Algorithm:

- Divide the interval [0,1] into 2M part of equal length $\Delta t = 1/(2M)$.
- Compute the G matrix by evaluate the integrals in equation (14).
- Compute the wavelet coefficients a_{r_i} by collocation method, using equation (15).
- Solve the linear system (16) by multiplication it with B^{-1}.

5. Numerical Examples:

In this section, we test some of the numerical examples performed in solving this linear system of Fredholm integral equations. The exact solution is used only to show the accuracy of the numerical solution obtained with our method.

Example 2: consider the problem:

$$u_1(x) = x^2 - \frac{4}{3}x^3 - \frac{9}{14} + \int_0^1 (x^3+2t)u_1(t)dt + \int_0^1 t^2 u_2(t)dt$$

$$u_2(x) = x^4 - \frac{4}{3}x^2 - \frac{1}{5}x + \frac{1}{7} + \int_0^1 x^2 u_1(t)dt + \int_0^1 (x-t^2)u_2(t)dt$$

with exact solution $u_1(x) = x^2 + 1 \quad , u_2(x) = x^4$ when J=2 we get the results in table (2)

x	Exact u_1	Haar wavelet method	Exact u_2	Haar wavelet method
٠,٠٦٢٥	1.0039062500	1.0135609855	0.0000152587	0.0007324117
٠,١٨٧٥	1.0351562500	1.0448814721	0.0012359619	0.0026920627
٠,٣١٢٥	1.0976562500	1.1076471520	0.0095367431	0.0120788023
٠,٤٣٧٥	1.1914062500	1.2019881543	0.0366363525	0.0406113806
٠,٥٦٢٥	1.3164062500	1.3280346080	0.1001129150	0.1058679226
٠,٦٨٧٥	1.4726562500	1.4859166420	0.2234039306	0.2312859282
٠,٨١٢٥	1.6601562500	1.6757643852	0.4358062744	0.4461622725
٠,٩٣٧٥	1.8789062500	1.8977079666	0.7724761962	0.7856532055
L.S.E.		٣,٥٣٥٠e-004		١,٧٣٦٣e-004

Table (2)**:** A comparison between the exact and numerical solution of Haar wavelet method when (J=2)

Computations were carried out for different values of J. These results were compared with the exact solution.

If we solve this example by trapezoidal method for different value of n and we compare these results with the Haar Wavelet method we get the Haar method is more reliable and accurate. The least square error of the obtained results for u_1 and u_2 are presented in table (3).

J	2M	L.S.E. for u_1 using Haar wavelet	L.S.E. for u_2 using Haar wavelet		N	L.S.E. for u_1 Using Trap.	L.S.E. for u_1 Using Trap.
1	4	0.00494572	0.002274896		4	0.022979765	0.012097952
2	8	٣,٥٣٥٠e-004	١,٧٣٦٣e-004		8	٠,٠٠١٦٧٠١٩٧	٨,٨٦٨٥e-004
3	16	2.4039e-005	1.2033e-005		16	1.0859e-004	5.7779e-005

Table (3) L.S.E. for u1 and u2 when using Haar wavelet method and trapezoidal method

Example 3: consider the problem:

$$u_1(x) = \sin(x) - \cos(1) + \sin(1) - x\sin(1) + \int_0^1 (x-t)u_1(t)dt + \int_0^1 xt\,u_2(t)dt$$

$$u_2(x) = \cos(x) - (1-\cos(1))x^2 + \cos(1) - 3\sin(1) - x\sin(1) + 1 + \int_0^1 (x^2+2t)u_1(t)dt + \int_0^1 (x+t)u_2(t)dt$$

with exact solution $u_1(x) = \sin(x) \quad , u_2(x) = \cos(x)$ when J=2 we get the results in table (4)

x	Exact u_1	Haar wavelet method	Exact u_2	Haar wavelet method
٠,٠٦٢٥	0.0624593178	0.0630779507	0.9980475107	0.9956415974
٠,١٨٧٥	0.1864032967	0.1870075309	0.9824733131	0.9797449294
٠,٣١٢٥	0.3074385145	0.3080283501	0.9515679480	0.9485442081
٠,٤٣٧٥	0.4236762572	0.4242516940	0.9058136834	0.9025217016
٠,٥٦٢٥	0.5333026735	0.5338637117	0.8459244992	0.8423913898
٠,٦٨٧٥	0.6346070800	0.6351537195	0.7728349461	0.7690878233
٠,٨١٢٥	0.7260086552	0.7265408960	0.6876855622	0.6837515402
٠,٩٣٧٥	0.8060811082	0.8065989504	0.5918050750	0.5877112683
L.S.E.		٢,٦٨١٦e-007		١,٦٧٥٩e-005

Table (4)**:** A comparison between the exact and numerical solution of Haar wavelet method when (J=2)

Computations were carried out for different values of J. These results were compared with the exact solution.

If we solve this example by trapezoidal method for different value of n and we compare these results with the Haar Wavelet method we get the Haar method is more reliable and accurate. The least square error of the obtained results for u_1 and u_2 are presented in table (5).

J	2M	L.S.E. for u_1 using Haar wavelet	L.S.E. for u_2 using Haar wavelet		N	L.S.E. for u_1 Using Trap.	L.S.E. for u_1 Using Trap.
1	4	4.3898e-006	2.6218e-004		4	1.7604e-005	0.0010715168
2	8	2.6816e-007	1.6759e-005		8	1.0595e-006	6.8450e-005
3	16	1.6552e-008	1.0616e-006		16	6.5552e-008	4.3026e-006

Table (5) L.S.E. for u1 and u2 when using Haar wavelet method and trapezoidal method

Example 4: consider the problem:

$$u_1(x) = e^x - e^1 x^3 + x^3 + 2e^{-1} + \int_0^1 (x^3 - t) u_1(t)dt + \int_0^1 t u_2(t)dt$$

$$u_2(x) = e^{-x} - e^1 x + 1 + xe^{-1} + \int_0^1 (x - t) u_1(t)dt + \int_0^1 x u_2(t)dt$$

with exact solution $u_1(x) = e^x \quad , u_2(x) = e^{-x}$

when J=2 we get the results in table (6)

x	Exact u_1	Haar wavelet method	Exact u_2	Haar wavelet method
٠,٠٦٢٥	1.0644944589	1.0680439134	0.9394130628	0.9402985481
٠,١٨٧٥	1.2062302494	1.2098002211	0.8290291181	0.8307382452
٠,٣١٢٥	1.3668379411	1.3704852470	0.7316156289	0.7341483977
٠,٤٣٧٥	1.5488302986	1.5526496335	0.6456485264	0.6490049370
٠,٥٦٢٥	1.7550546569	1.7591785937	0.5697828247	0.5739628770
٠,٦٨٧٥	1.9887374695	1.9933364590	0.5028315779	0.5078352720
٠,٨١٢٥	2.2535347872	2.2588171580	0.4437473100	0.4495746459
٠,٩٣٧٥	2.5535894580	2.5598014169	0.3916056266	0.3982566043
L.S.E.		٣,٨٥٨٨e-005		٤,٤٢٣٥e-005

Table (6)**:** A comparison between the exact and numerical solution of Haar wavelet method when (J=2)

Computations were carried out for different values of J. These results were compared with the exact solution.

If we solve this example by trapezoidal method for different value of n and we compare these results with the Haar Wavelet method we get the Haar method is more reliable and accurate. The least square error of the obtained results for u_1 and u_2 are presented in table (7).

J	2M	L.S.E. for u_1 using Haar wavelet	L.S.E. for u_2 using Haar wavelet		N	L.S.E. for u_1 Using Trap.	L.S.E. for u_1 Using Trap.
1	4	5.2068e-004	6.2740e-004		4	0.002921794	0.003063933
2	8	3.8588e-005	4.4236e-005		8	1.8379e-004	1.9715e-004
3	16	2.6329e-006	2.9334e-006		16	1.1505e-005	1.2411e-005

Table (7) L.S.E. for u1 and u2 when using Haar wavelet method and trapezoidal method

Conclusion:

The main aim of the present paper is to propose for numerical solution of linear system of Fredholm integral equations by a simple method based on the Haar wavelets. Lease square error show that the accuracy of computations is very high even when the mode number is small. The method is computationally efficient and the algorithm can easily be implemented on a computer. Numerical comparisons demonstrate that our method is more reliable and accurate.

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الحل العددي لمنظومة خطية من معادلات فريدهوم التكاملية باستخدام طريقة مويجات هار

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الخلاصة:

الهدف من البحث هو تقديم طريقة عددية لحل منظومة خطية من معادلات فريدهولم التكاملية بالاعتماد على طريقة المويجات هار .

اختبرنا العديد من الامثلة بمقارنة الحل العددي مع الحل المظبوط. كذلك قارنا نتائج الطريقة المقترحة في هذا البحث مع نتائج طرق أخرى مثل طريقة شبه المنحرف . الخوارزمية والبرنامج كتبت ببرنامج(MATLAB (version 7.0)).

الكلمات المفتاحية : منظومة معادلات فريدهولم التكاملية الخطية ، طريقة مويجات هار.