The Artin Exponent of Projective Special Linear Group PSL (2, P^k)

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Abstract:

This paper finds the Artin characters and Artin exponent depending on the character table and conjugacy classes of projective special linear group PSL (2,P^k).Then we prove that Artin exponent of PSL (2, P^k) is equal to $\mathbf{p^{k-1}}$ where P is a prime number , p=3 and k> 0.

Key words : Character table, Artin character, Artin exponent, PSL (2, P^k).

1. Introduction:

The Artin Exponent induced from cyclic subgroups of finite groups was studied extensively by Lam.T in [7]. A Burnside ring theoretic version of the results in [lam] for p- groups was given in [3].Here we shall be interested in looking at the Artin exponent induced from the cyclic subgroups of finite projective special linear group (2, p^k).

After we construct the ordinary character table of the finite projective special linear groups is derived as well.

In section two we take a further step to find the Artin character and Artin exponent of projective special linear group PSL (2, p^k) where p=3, k> 0.

In section three we take some particular examples.

2. Basic Concepts and Theorems

In this section the mian information of PSL (2, p^k) are introduced:

Definition 1.1[8,2] : The projective general linear group PGL (n, f) an projective special linear group PSL (n, f) are the quotients of GL (n, f) and SL (n, f) respectively.

$$PGL\ (n,f) = \frac{GL\ (n,f)}{Z(GL\ (n,f))} \qquad ;\ PSL\ (n,f) = \frac{SL\ (n,f)}{Z(SL\ (n,f))}$$

Theorem 1.2[1,6] :

(i) The group $PSL(2,p^k)$ is simple for $p^k > 3$.

(ii) $|PSL(2,p^k)| = \begin{cases} (p^k+1)p^k(p^k-1) \text{ if } p = 2; \\ \ldots \end{cases}$

Lemma1.3[1,2]: PSL(2, P^k) has exactly $(2P^k+10)/4$ conjugacy classes $C_{(z)g} \in PSL(2,P^k)$. For $P^k \equiv +1$ (mod 4):

Table(1):

$<z>g$	C_g	$\|C_g\|$	$\|C_G(g)\|$
$<z>$	$C_{<z>}$	1	$P^k(P^{2k}-1)/2$
$<z>c$	$C_{<z>c}$	$(P^{2k}-1)/2$	P^k
$<z>d$	$C_{<z>d}$	$(P^{2k}-1)/2$	P^k
$<z>a^l$	$C_{<z>a^1}$	$P^k(P^k+1)$	$(P^k-1)/2$
$<z>a^{(P^k-1)/4}$	$C_{<z>a^{(P^k-1)/4}}$	$P^k(P^k+1)/2$	(P^k-1)
$<z>b^m$	$C_{<z>b^m}$	$P^k(P^k-1)$	$(P^k+1)/2$

where $1 \le l \le (P^k-5)/4$ and $1 \le m \le (P^k-1)/4$

For $P^k \equiv -1$ (mod 4):

Table (2):

$<z>g$	C_g	$\|C_g\|$	$\|C_G(g)\|$
$<z>$	$C_{<z>}$	1	$P^k(P^{2k}-1)/2$
$<z>c$	$C_{<z>c}$	$(P^{2k}-1)/2$	P^k
$<z>d$	$C_{<z>d}$	$(P^{2k}-1)/2$	P^k
$<z>a^l$	$C_{<z>a^l}$	$P^k(P^k+1)$	$(P^k-1)/2$
$<z>b^m$	$C_{<z>b^m}$	$P^k(P^k-1)$	$(P^k+1)/2$
$<z>b^{(P^k+1)/4}$	$C_{<z>b^{(P^k-1)/4}}$	$P^k(P^k+1)/2$	(P^k-1)

where $1 \le l \le (P^k-3)/4$ and $1 \le m \le (P^k-3)/4$

3. The mian results

In this section we shall give our results about the Artin exponent a(G) of the finite special linear group PSL(2,P^k), k is natural number, k>0.

Theorem 3.1

Let G=PSL(2,P^k), k=natural, k>0. Then a(G)=3^{k-1} and the table of characters induced from the characters of all its cyclic subgroups.

For $P^{k}\equiv +1$ (mod 4)

Table (3):

$<z>g$	$<z>$	$<z>C$	$<z>d$	$<z>a(P^k-1)/4$	a^ℓ	b^m
$\lvert C_{(g)}\rvert$	1	$(P^{2k}-1)/2$	$(P^{2k}-1)/2$	$P^k(P^k+1)/2$	$P^k(P^k+1)$	$P^k(P^k-1)$
$\lvert C_G(g)\rvert$	$P^k(P^{2k}-1)/2$	P^k	P^k	(P^k-1)	$(P^k-1)/2$	$(P^k+1)/2$
Q_1	$P^k(P^{2k}-1)/2$	0	0	0	0	0
Q_2	$P^k(P^{2k}-1)/2P$	$-P^k/P$	0	0	0	0
Q_3	$P^k(P^{2k}-1)/2P^2$	0	$-P^k/P^2$	0	0	0
Q_4	$P^k(P^{2k}+1)/2$	0	0	-1	0	0
Q_5	$\ell(P^k(P^k+1))$	0	0	0	-1	0
Q_6	$M(P^k(P^k-1))$	0	0	0	0	-1

where $1\le\ell\le(P^k-5)/4$ and $1\le m\le(P^k-1)/4$

Proof:

$|PSL(2,P^k)| = P^k(P^{2k}-1)$ (by lemma (1.2))

From theorem (1.3), $G=PSL(2,3^k)$ has exactly (3^k+1) cojugacy classes C_g for $g\in G$.

For$P^k\equiv +1 \pmod 4$

Table (4):

$<z>g$	$<z>$	$<z>C$	$<z>d$	$<z>a(P^k-1)/4$	a^ℓ	b^m
$\lvert C_{(g)}\rvert$	1	$(P^{2k}-1)/2$	$(P^{2k}-1)/2$	$P^k(P^k+1)/2$	$P^k(P^k+1)$	$P^k(P^k-1)$
$\lvert C_G(g)\rvert$	$P^k(P^{2k}-1)/2$	P^k	P^k	(P^k-1)	$(P^k-1)/2$	$(P^k+1)/2$

where: $1\le\ell\le(P^k-5)/4$ and $1\le m\le(P^k-1)/4$

By the definition of inducing we obtained the induced characters Q_1, Q_2, Q_3, Q_4, Q_5 and Q_6ofPSL(2,P^k) from the characters of all cyclic subgroups[9].

Table (5):

$<z>g$	$<z>$	$<z>C$	$<z>d$	$<z>a(P^k-1)/4$	a^ℓ	b^m
Q_1	$P^k(P^{2k}-1)/2$	0	0	0	0	0
Q_2	$P^{k-1}(P^{2k}-1)/2$	$-P^{k-1}$	0	0	0	0
Q_3	$P^{k-2}(P^{2k}-1)/2$	0	$-P^{k-2}$	0	0	0
Q_4	$P^k(P^k+1)/2$	0	0	-1	0	0
Q_5	$\ell(P^k(P^k+1))$	0	0	0	-1	0
Q_6	$m(P^k(P^k-1))$	0	0	0	0	-1

Then we have table (3)

By multiply Q_6 by -1 we get: $-m(P^k(P^k-1))$, by multiply Q_5 by -1 we get: $-\ell(P^k(P^k+1))$

By multiply Q_4 by -1 we get: $-P^k(P^{2k}+1)/2$, by multiply Q_3 by -1 we get: $-P^k(P^{2k}-1)/2P^2$

By multiply Q_2 by -1 we get: $-P^k(P^{2k}-1)/2P$, and then adding the result to $Q_1=P^k(P^{2k}-1)/2$ we get:

$$\frac{-P^k}{4}(P^k-1)(P^k+1)-\frac{P^k}{4}(P^k-5)(P^k-1)-\frac{P^k}{2}(P^k+1)-(P^{2k}-1)+\frac{P^k}{2}(P^{2K}-1)$$

$$=\frac{-P^k}{4}(P^k-1)(P^k+1+P^k-5)-\frac{P^k}{2}(P^k+1)-(P^{2k}-1)+\frac{P^k}{2}(P^{2k}-1)$$

$$=\frac{-P^k}{4}(P^k-1)(2P^k-4)-\frac{P^k}{2}(P^k+1)-(P^{2k}-1)+\frac{P^k}{2}(P^{2k}-1)$$

$$=\frac{-P^k}{2}(P^k-1)(P^k-2)-\frac{P^k}{2}(P^k+1)-(P^{2k}-1)+\frac{P^k}{2}(P^{2k}-1)$$

$$=\frac{-P^k}{2}(P^{2k}-3P^k-2)-\frac{P^{2k}}{2}-\frac{P^k}{2}-P^{2k}+1+\frac{P^{3k}}{2}-\frac{P^k}{2}$$

$$=\frac{-P^{3k}}{2}+\frac{3}{2}P^{2k}+P^k-\frac{P^{2k}}{2}-\frac{P^k}{2}-P^{2k}+\frac{P^{3k}}{2}-\frac{P^k}{2}+1=1$$

Theorem 3.2

Let $G=PSL(2,P^k)$, k=natural, k>0. Then $a(G)=3^{k-1}$ and the table of characters induced from the characters of all its cyclic subgroups.

For $P^k \equiv -1 \pmod 4$

Table (6)

<z>g	<z>	<z>C	<z>d	$b(P^k+1)/4$	a^ℓ	b^m
$\lvert C_{(g)}\rvert$	1	$(P^{2k}-1)/2$	$(P^{2k}-1)/2$	$P^k(P^k-1)/2$	$P^k(P^k+1)$	$P^k(P^k-1)$
$\lvert C_G(g)\rvert$	$P^k(P^{2k}-1)/2$	P^k	P^k	(P^k+1)	$(P^k-1)/2$	$(P^k+1)/2$
Q_1	$P^k(P^{2k}-1)/2$	0	0	0	0	0
Q_2	$P^k(P^{2k}-1)/2$	$-P^{k-1}$	0	0	0	0
Q_3	$P^k(P^{2k}-1)/2$	0	$-P^{k-2}$	0	0	0
Q_4	$P^k(P^{2k}+1)/2$	0	0	-1	0	0
Q_5	$\ell(P^k(P^k+1))$	0	0	0	-1	0
Q_6	$m(P^k(P^k-1))$	0	0	0	0	-1

where $1\le\ell\le(P^k-3)/4$ and $1\le m\le(P^k-3)/4$

Proof:

$|PSL(2,P^k)| = P^k(P^{2k}-1)$ (by lemma (1.2))

From theorem (1.3), $G=PSL(2,3^k)$ has exactly (3^k+1) cojugacy classes C_g for $g\in G$.

For $P^k \equiv -1 \pmod 4$

Table (7)

<z>g	<z>	<z>C	<z>d	$b(P^k+1)/4$	a^ℓ	b^m
$\lvert C_{(g)}\rvert$	1	$(P^{2k}-1)/2$	$(P^{2k}-1)/2$	$P^k(P^k-1)/2$	$P^k(P^k+1)$	$P^k(P^k-1)$
$\lvert C_G(g)\rvert$	$P^k(P^{2k}-1)/2$	P^k	P^k	(P^k+1)	$(P^k-1)/2$	$(P^k+1)/2$

where: $1\le\ell\le(P^k-3)/4$ and $1\le m\le(P^k-3)/4$

By the definition of inducing we obtained the induced characters Q_1, Q_2, Q_3, Q_4, Q_5 and Q_6 of PSL(2,P^k) from the characters of all cyclic subgroups.

Table (8)

<z>g	<z>	<z>C	<z>d	$b(P^k+1)/4$	a^ℓ	b^m
Q_1	$P^k(P^{2k}-1)/2$	0	0	0	0	0
Q_2	$P^k(P^{2k}-1)/2$	$-P^{k-1}$	0	0	0	0
Q_3	$P^k(P^{2k}-1)/2$	0	$-P^{k-2}$	0	0	0
Q_4	$P^k(P^{2k}+1)/2$	0	0	-1	0	0
Q_5	$\ell(P^k(P^k+1))$	0	0	0	-1	0
Q_6	$m(P^k(P^k-1))$	0	0	0	0	-1

Then we have table (6)

By multiply Q_6 by -1 we get: $-m(P^k(P^k-1))$, by multiply Q_5 by -1 we get: $-\ell(P^k(P^k+1))$,

by multiply Q_4 by -1 we get: $-P^k(P^{2k}+1)/2$, by multiply Q_3 by -1 we get: $-P^k(P^{2k}-1)/2$,

by multiply Q_2 by -1 we get: $-P^k(P^{2k}-1)/2$, and then adding the result to $Q_1=P^k(P^{2k}-1)/2$ we get:

$$\frac{-P^k}{4}(P^k-3)(P^k-1)-\frac{P^k}{4}(P^k-3)(P^k+1)-\frac{P^k(P^k-1)}{2}-\frac{(P^{2k}-1)}{2}+\frac{P^k(P^{2k}-1)}{2}$$

$$=\frac{-P^k}{4}(P^k-3)(2P^k)-\frac{P^k}{2}(P^k-1)+\frac{P^k}{2}(P^{2k}-1)-(P^{2k}-1)$$

$$=\frac{-P^{2k}}{2}(P^k-3)-\frac{P^k}{2}(P^k-1)+\frac{P^{3k}}{2}-\frac{P^k}{2}-P^{2k}+1$$

$$=\frac{-P^{3k}}{2}+\frac{3P^{2k}}{2}-\frac{P^{2k}}{2}+\frac{P^k}{2}+\frac{P^{3k}}{2}-\frac{P^k}{2}-P^{2k}+1=1$$

4. Some Examples

To motivate the general algebraic procedure we take some particular examples :

1) $|PSL(2,3)|=\frac{1}{2}P^K(P^K-1)=\frac{1}{2}3(8)=12$

The cojugacy classes is $\frac{2P^K+10}{4}=\frac{16}{4}=4$, For $P^K\equiv -1$ (mod4)

$\Longrightarrow$ Artin's character are

Table (9):

$<z>g$	$<z>$	$<z>C$	$<z>d^1$	b^1
$\lvert C_g\rvert$	1	4	4	3
$\lvert C_G(g)\rvert$	12	3	3	4
Q_1	12	0	0	0
Q_2	4	-1	0	0
Q_3	4	0	-1	0

Q_4	3	0	0	-1

where $\ell = 1$, $m = 1$

$\Rightarrow -Q_4$	-3	0	0	1
$-Q_3$	-4	0	1	0
$-Q_2$	-4	1	0	0
	-11	1	1	1
$+Q_1$	12	0	0	0
	1	1	1	1

$a(PSL\ (2,3)) = 1 = 3 = 3^{k-1}$

2) $|PSL(2,3^2)| = \frac{1}{2}9(80) = 360$

The cojugacy classes are = $\frac{2*9+10}{4} = \frac{28}{4} = 7$ For $P^K \equiv +1$ (mod 4)

$\Rightarrow$ Artin's characters are

Table (10):

$<z>g$	$<z>$	$<z>C$	$<z>d$	$<z>a^2$	a^1	b^1	b^2
$\|C_g\|$	1	40	40	45	90	72	72
$\|C_G(g)\|$	360	9	9	4	8	5	5
Q_1	360	0	0	0	0	0	0
Q_2	120	-3	0	0	0	0	0
Q_3	40	0	-1	0	0	0	0
Q_4 Q_5 Q_6	45 90 144	0 0 0	0 0 0	-1 0 0	0 -1 0	0 0 -1	0 0 -1

where $\ell = 1$, $m = 2$

$\Rightarrow -Q_6 =$	-144	0	0	0	0	-1	1
$-Q_5 =$	-90	0	0	0	1	1	1
$-Q_4 =$ $-Q_3 =$ $-\frac{1}{3}Q_2 =$	-45 -40 -40	0 0 1	0 1 0	1 1 0	1 1 0	1 1 0	1 1 0
$+Q_1$	-359 360	1 0	1 0	1 0	1 0	1 0	1 0

3) $|PSL\ (2,3^7)| = \frac{2187\ (4782968)}{2} = 5930175508$ Cons classes = $\frac{2*2187+10}{4}$ = 1096

For $3^7 \equiv -1$ mode 4

where $1 \le \ell \le \frac{2184}{4} = 546$, $1 \le m \le \frac{2184}{4} = 546$, $<z>b^{547}$

Table (11):

<z>g	<z>	<z>C	<z>d	<z>b^{547}	a^1………	a^{546}	b^1………	b^{546}
$\lvert C_g \rvert$	1	2391484	2391484	2390391	4785156………	4785156	4780782………	4780782
$\lvert C_G(g) \rvert$	5230175508	2187	2187	2188	1093………	1093	1094………	1094
Q_1	5230175508	0	0	0	0………	0	0………	0
Q_2	1743391836	-729	0	0	0………	0	0………	0
Q_3	581130612	0	-243	0	0………	0	0………	0
Q_4	2390391	0	0	-1	0………	0	0………	0
Q_5	2612695176	0	0	0	-1………	-1	0………	0
Q_6	2610306972	0	0	0	0………	0	-1………	-1
$-Q_6$	-2610306972	0	0	0	0………………0		1………………1	
$-Q_5$	-2612695176	0	0	0	1………………1		0………………0	
$-Q_4$	-2390391	0	0	1	0………………0		0………………0	
$\frac{-1}{243}Q_3$	$(\frac{-1}{243})$ 581130612	0	$(\frac{243}{243})$	0	0………………0		0………………0	
$\frac{-1}{729}Q_2$	$(\frac{-1}{729})$ 1743391836	$(\frac{729}{729})$	0	0	0………………0		0………………0	
$\Rightarrow$ $+Q_1$	-5230175507 +5230175508	1 0	1 0	1 0	1………………1 0………………0		1………………1 0………………0	
	1	1	1	1	1………………1		1………………1	

$$\Rightarrow a(PSL(2,P^7)) = 729 = 3^6 = 3^{K-1}$$

5.Discussions

The Artin exponent and Artin characters for several groups of PSL (2, 3^k) are calculated , and we find that:

For k=1, For $P^K \equiv -1$ (mod4), Artin exponent of PSL (2, 3) =1=3^0=3^{k-1}

For k=2, For $P^K \equiv +1$ (mod4), Artin exponent of PSL (2, 3^2) =3=3^1=3^{k-1}

For k=3, For $P^K \equiv -1$ (mod4) ,Artin exponent of PSL (2, 3^3) =9=3^2=3^{k-1}

For k=4, For $P^K \equiv +1$ (mod4) ,Artin exponent of PSL (2, 3^4) =27=3^3=3^{k-1}

For k=5, For $P^K \equiv -1$ (mod4), Artin exponent of PSL (2, 3^5) =81=3^4=3^{k-1}

For k=6, For $P^K \equiv +1$ (mod4), Artin exponent of PSL (2, 3^6) =243=3^5=3^{k-1}

For k=7, For $P^K \equiv -1$ (mod4) ,Artin exponent of PSL (2, 3^7) =729=3^6=3^{k-1}

Hence, in general Artin exponent of projective special linear group PSL (2, p^k) where p=3, k> 0 is equal to 3^{k-1} .

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أس ارتن للزمر الخطية الخاصة الأسقاطية PSL (2, P^k)

المستخلص:

في هذا البحث تم إيجاد رمز ارتن وأس ارتن بالاعتماد على جدول الرمز العام والمجموعات الجزئية للزمر الخطية الخاصة الاسقاطية PSL(2,P^k)، ثم برهنا أن أس ارتن للزمر الخطية الخاصة الأسقاطية هو PSL (2, P^k)= p^{k-1} حيث إن P عدد أولي ويساوي 3, و K> 0.