Wavelet Approach to Estimate The solution for Two Points Boundary Value Problems

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Abstract

In this paper, a new algorithm based on using Chebyshev wavelet operational matrix of integration instead of operational matrix of derivative to find the approximate solution of two points boundary value problem. Wavelet first Chebyshev with aid of collocation method employed to transform the differential equation with its boundary condition to a system of linear algebraic equations. A very high level of accuracy explicitly reflected by the proposed examples.

Keyword: *Chebyshev wavelets, two points boundary value problem.*

1. Introduction:

The solution of two points linear boundary value problems (TPLBVP)or multi-points problems have major role in the fields of science and engineering when a physical system is modeled under boundary value problem in two differential points.

The vibration of a guy wire of uniform cross-section composed of N parts of differential densities can be set up as a multi- points BVP ,also many problems in the theory of elastic stability can be handled by the method of two points boundary problems[1].

In this paper, we consider linear two points boundary value problems of the form

$$y^{(n)}(x) = g(x) + f(y) \qquad 0 \le x \le 1 \qquad \ldots \quad (1)$$

with

$$y^{(n-1)}(0) = \alpha_{n-1} \quad , y(1) = \beta_0 \quad , y'(1) = \beta_1 \quad , n = 1,2,... \qquad \ldots \quad (2)$$

Where $g(x)$ is a source term function, $f(y)$ is a given continuous linear function and α_{n-1} and β_0, β_1 are real finite constants.

A substantial amount of research work has been done for the study of two or multipoint boundary value problems to obtain higher accuracy rapidly by using wavelet basis;such that Elhameed.W.M [1] employed third and fourth wavelet Chebyshev with aid of collocation method for solving second order multipoint BVP,also Ayyaz [3] applied second wavelet

Chebyshev to solve fifth and sixth order boundary value problems .Fazal [7] proposed uniform Haar wavelets for the numerical solution of sixth order two points boundary value problems. The main aim of this paper is to develop a new collocation algorithm for solving two-point boundary value problem based on first kind Chebyshev wavelets defined in four sub intervals on the interval [0,1]. Operational matrix of integration derived. The method reduces the differential equation with its initial and boundary conditions to a system of algebraic equations in the unknown expansion coefficients.

2. Chebyshev Wavelets Preliminaries [2,3,5,6,9]:

Wavelets constitute of a family of functions constructed from dilation and translation of single function called the mother wavelet. When the dilation parameter (a) and the translation parameter (b) varies continuously, then we have the following family of continuous wavelets:

$$\psi_{a,b}(t)=|a|^{-1/2}\psi\left(\frac{t-b}{a}\right) \qquad a,b\in R \qquad a\neq 0 \qquad \ldots(3)$$

First kind Chebyshev wavelets $\Psi nm(t)=\psi(k,n,m,t)$ have four arguments: k,n can assume any positive integer, m is the order of first kind Chebyshev polynomials, and t is the normalized time, they are defined on the interval [0,1] by

$$\psi_{n,m}(t)=\begin{cases}\dfrac{\alpha_m 2^{k/2}}{\sqrt{\pi}}T_m(2^{k+1}t-2n+1) & \dfrac{n-1}{2^k}\le t\le\dfrac{n}{2^k}\\ 0 & otherwise\end{cases} \qquad \ldots(4)$$

Where

$$\alpha_m=\begin{cases}\sqrt{2} & m=0\\ 2 & m=1,2,..\end{cases}$$

Where $T_m(t)$ the first kind Chebyshev polynomials of order m .which are orthogonal with respect to the weight function $w(t)=\dfrac{1}{\sqrt{1-t^2}}$ and satisfy the following recurrence formula

$$T_0=1\ ,\ T_1(t)=t\ ,\ T_{m+1}(t)=2t\,T_m(t)-T_{m-1}(t) \quad m=1,2,3,... \qquad \ldots(5)$$

The set of Chebyshev wavelets are an orthogonal set with respect to the weight function

$$w_n(t)=w(2^{k+1}t-2n+1)$$

Chebyshev wavelet scaling functions can be obtained when $(M=3,k=2)$ from eq (4) as follows:

$$\left.\begin{aligned} \psi_{10}(t) &= \sqrt{\frac{8}{\pi}} \\ \psi_{11}(t) &= \frac{4}{\sqrt{\pi}}(8t-1) \\ \psi_{12}(t) &= \frac{4}{\sqrt{\pi}}(2(8t-1)^2-1) \end{aligned}\right\} 0 \le t \le 1/4$$

$$\left.\begin{aligned} \psi_{20}(t) &= \sqrt{\frac{8}{\pi}} \\ \psi_{21}(t) &= \frac{4}{\sqrt{\pi}}(8t-3) \\ \psi_{22}(t) &= \frac{4}{\sqrt{\pi}}(2(8t-3)^2-1) \end{aligned}\right\} 1/4 \le t \le 1/2 \qquad \ldots (6)$$

$$\left.\begin{aligned} \psi_{30}(t) &= \sqrt{\frac{8}{\pi}} \\ \psi_{31}(t) &= \frac{4}{\sqrt{\pi}}(8t-5) \\ \psi_{32}(t) &= \frac{4}{\sqrt{\pi}}(2(8t-5)^2-1) \end{aligned}\right\} 1/2 \le t \le 3/4$$

$$\left.\begin{aligned} \psi_{40}(t) &= \sqrt{\frac{8}{\pi}} \\ \psi_{41}(t) &= \frac{4}{\sqrt{\pi}}(8t-7) \\ \psi_{42}(t) &= \frac{4}{\sqrt{\pi}}(2(8t-7)^2-1) \end{aligned}\right\} 3/4 \le t \le 1$$

A function $f(t)$ defined over [0,1] may be expanded as

$$f(t) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} C_{ij}\psi_{ij}(t) \qquad \ldots(7)$$

where

$$C_{ij} = (f(t), \psi_{ij}(t))w = \int_0^1 w(t)\psi_{ij}(t)f(t)dt \qquad \ldots(8)$$

in which $\langle .,. \rangle$ denoted the inner product in $L^2 w[0,1]$. If the infinite series in eq(7) is truncated ,then it can be written as

$$f(t) = \sum_{i=1}^{2^k}\sum_{j=0}^{m} C_{ij}\psi_{ij}(t) = C^T\psi(t) \qquad \ldots(9)$$

where C and $\psi(t)$ are $2^k M \times 1$ matrices given by

$$C = [c_{10}, c_{11}, ..., c_{1M-1}, c_{20}, c_{21}, ..., c_{2M-1}, c_{30}, c_{31}, ..., c_{3M-1}, c_{40}, c_{41}, ..., c_{4M-1}]^T \qquad \ldots(10)$$

and

$$\psi(t) = [\psi_{10}, \psi_{11}, ..., \psi_{1M-1}, \psi_{20}, \psi_{21}, ..., \psi_{2M-1}, \psi_{30}, \psi_{31}, ..., \psi_{3M-1}, \psi_{40}, \psi_{41}, ..., \psi_{4M-1}]^T \quad ..(11)$$

3. *Chebyshev wavelets operational matrix of integration* [2,4,5,7,8,9]:

In this section we will derive the operational matrix P of integration which plays a great role in dealing with our method for solving TPLBVP. For Chebyshev wavelet the integration of the vector $\psi(t)$ defined in eq(11) can be obtained by

$$\int_0^t \psi(s)ds \cong P\psi(t) \qquad \ldots(12)$$

where P is the $(2^k M)\times(2^k M)$ operational matrix for integration and is given as

$$P=\begin{bmatrix} C & S & S & \cdots & S \\ 0 & C & S & \cdots & S \\ 0 & 0 & C & \cdots & S \\ \vdots & \vdots & \vdots & \ddots & S \\ 0 & 0 & 0 & \cdots & C \end{bmatrix} \qquad \ldots(13)$$

where C and S are m×m matrices given by:

$$C=\frac{1}{2^k}\begin{bmatrix} \frac{1}{2} & \frac{1}{2\sqrt{2}} & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{4\sqrt{2}} & 0 & \frac{1}{8} & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{3\sqrt{2}} & \frac{-1}{4} & 0 & \frac{1}{12} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{-1}{2\sqrt{2}(M-1)(M-3)} & 0 & 0 & 0 & \cdots & \frac{-1}{4(M-3)} & 0 & \frac{-1}{4(M-1)} \\ \frac{-1}{2\sqrt{2}M(M-2)} & 0 & 0 & 0 & \cdots & 0 & \frac{-1}{4(M-2)} & 0 \end{bmatrix} \qquad \ldots(14)$$

$$S=\frac{\sqrt{2}}{2^k}\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{-1}{3} & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{-1}{15} & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{-1}{M(M-2)} & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \qquad \ldots(15)$$

4. *Chebyshev wavelets collocation method*

In this section, the introduced Chebyshev wavelets will be applied with aid of collocation method to solve TPLBVP of order n.

Consider the TPLBVP of order n

$$y^{(n)}(x)=g(x)+f(y) \qquad \ldots(16)$$

with the boundary conditions

$$y^{(n-1)}(0)=\alpha_{(n-1)}\ ,y(1)=\beta_0\ ,y'(1)=\beta_1 \qquad n=1,2,...$$

A function $y^{(n)}(x)$ can be expanded approximately using chebyshev wavelet series as

$$y^{(n)}(x)=\sum_{i=1}^{2^k}\sum_{j=0}^{m}C_{ij}\psi_{ij}(x) \qquad \ldots(17)$$

where C_{ij} are the wavelets coefficients integrate eq(17) from 0 to t , yields

$$y^{(n-1)}(t)-y^{(n-1)}(0)=\int_0^t C^T\psi(s)\,ds=C^T P\psi(t)$$

where P is the operational matrix of integration that is

$$y^{(n-1)}(t)=y^{(n-1)}(0)+C^T P\psi(t) \qquad ...(18)$$

integrate eq(18), $(n-1)$ times from o to t ,one can obtain

$$y^{(n-2)}(t)=y^{(n-2)}(0)+y^{(n-1)}(0)t+C^T P^2\psi(t) \qquad \ldots(19)$$

$$\vdots$$

$$y(t)=y(0)+y'(0)t+\frac{y''(0)}{2!}t^2+...+\frac{y^{(n-1)}(0)}{(n-1)!}t^{(n-1)}+C^T P^n\psi(t) \qquad ...(20)$$

.

substituting eq(17) and eq(20) into eq(16) we have

$$\sum_{i=1}^{2^k}\sum_{j=0}^{m}C_{ij}\psi_{ij}(t)=g(x)+(C^T P^n\psi(t)+y(0)+y'(0)t+\frac{y''(0)t^2}{2!}+...+\frac{y^{(n-1)}(0)t^{(n-1)}}{(n-1)!}) \qquad ...(21)$$

Then a total number of 2^{k-1}M conditions should exist for determination of 2^{k-1}M coefficients

$$c_{10},c_{11},...,c_{2^kM-1},c_{20},...,c_{2^kM-1},c_{30},c_{31},...,c_{2^KM-1},c_{40},c_{41},...,c_{2^KM-1}$$

since numbers of conditions furnished by some gives boundary conditions, at (t=0 and t=1), namely

$$y(t)=C^T P^n\psi(t)+y(0)+y'(0)t+...+\frac{y^{(n-1)}t^{(n-1)}}{(n-1)!} \qquad \ldots(22)$$

$$\frac{dy(t)}{dt}=C^T P^{n-1}\psi(t)+\sum_{z=0}^{n-2}\frac{y^{(z)}(0)t^z}{z!} \qquad \ldots(23)$$

we see that there should be (2^{k-1}M)-S, (S is the number of given boundary conditions) extra conditions to recover the unknown coefficients c_{ij}. These conditions can be obtained by applying eq(21) at the collocation point

$t_i=\dfrac{i}{(2^kM)-S} \qquad i=1,2,3,...$ where (2^{k-1}M)-S≠0

Combine equations (21),(22),and (23) to obtain ($2^{k-1}M$) linear equations form which we can compute values for the unknown coefficients C_{ij} .

5. Numerical Examples:

For showing efficiency of our approximate method, we consider the following examples.

***Example* (1):**

Consider the following linear two point boundary value problem:

$$y''-y=2-t^2 \qquad 0\le t\le 1$$

Subject to the boundary conditions

$$y(0)=y'(0)=0,\, y(1)=1,\, y'(1)=2\,,$$

The exact solution of this problem $y(t)=t^2$.

For solving this problem using Chebyshev wavelet collocation method:

Assume that

$$y''(t)\cong\sum_{i=1}^{2^k}\sum_{j=0}^{M-1}c_{ij}\psi_{ij} \qquad \ldots(24)$$

where ψ_{ij} is the first wavelet Chebyshev .

So

$$Y(t)\cong P^2C^T\psi(t)$$

By integrating eq(24) two times and use operational matrix of integral to applying the steps studied in section(4) the approximate coefficients will be:

$c_{10}=1.2533,\qquad c_{11}=2.7375e-013\,,\qquad c_{12}=-3.2255e-005,$

$c_{20}=1.2531,\qquad c_{21}=-8.6282e-005,\qquad c_{22}=-1.4804e-004,$

$c_{30}=1.2533,\qquad c_{31}=-1.0124e-005,\qquad c_{32}=-1.2500e-005,$

$c_{40}=1.2548,\qquad c_{41}=0.0011,\qquad c_{42}=0.0014$

Substituting the approximate coefficients in eq(24) hence the approximate solution is:

$y(t)\cong t^2$ .which is the exact solution.

***Example* (2)**:

Consider the following linear two point boundary value problem:

$$y^{(4)}(t)+y=2e^t \qquad 0\le t\le 1$$

Subject to the boundary conditions

$$y(0)=y'(0)=y^k(0)=1 \quad k=0,1,..,3$$

The exact solution of this problem $y(t)=e^t$.

Assume that

$$y^{(4)}(t)\cong\sum_{i=1}^{2^k}\sum_{j=0}^{M-1}c_{ij}\psi_{ij} \qquad \ldots(25)$$

So

$Y(t) \cong P^4 C^T \psi(t)$

Integrating eq(25) four times and using operational matrix of integration to applying the steps studied in section(4). Table (1) shows the comparison of the absolute error between exact solution and approximate results for m=3, K=2.

Table (1): Approximate results of example 2.

T	Exact Solution	Approximate Solution	Absolute Error
0.0	1.0000	1.0000	1.1016e-005
0.1	1.1052	1.1052	1.1509e-005
0.2	1.2214	1.2214	8.6437e-006
0.3	1.3498	1.3498	3.6931e-005
0.4	1.4918	1.4918	1.3376e-005
0.5	1.6488	1.6487	7.0599e-005
0.6	1.8221	1.8221	4.6319e-005
0.7	2.0138	2.0138	7.9291e-005
0.8	2.2255	2.2255	6.5458e-005
0.9	2.4597	2.4596	1.3593e-004
1.0	2.7183	2.7183	8.8818e-016

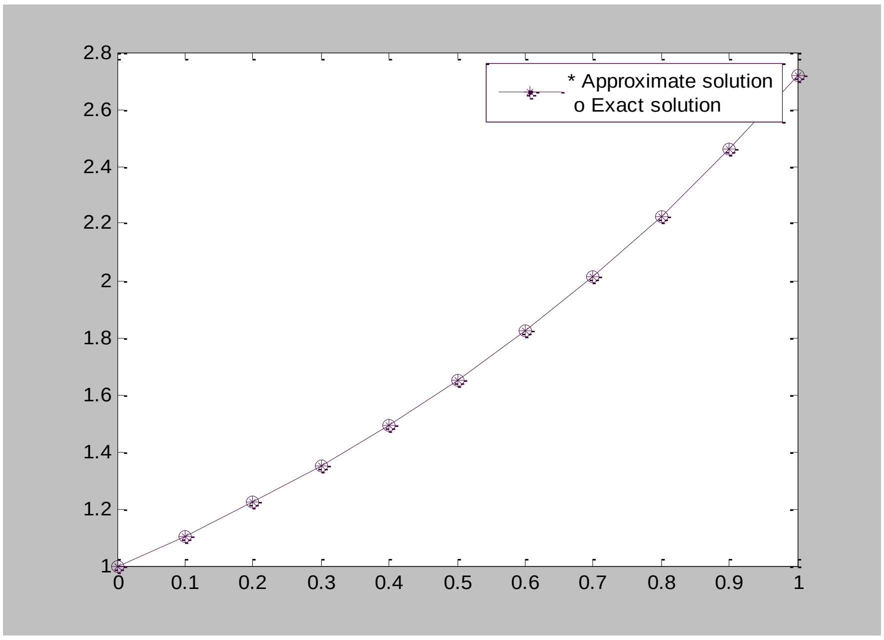

Figure (1): the plot of exact solution and approximate results using Chebyshev wavelets collocation method when m=3,K=2 for example 2.

***Example* (3):[7]**

Consider the following linear two points boundary value problem:

$$y^{(5)}(t) = y - 15e^{t} - 10te^{t} \qquad 0 \le t \le 1$$

Subject to the boundary conditions

$$y(0)=0,\ y'(0)=1,\ y''(0)=0,\ y^{(3)}(0)=-3,\ y^{(4)}(0)=-8$$

$$y(1)=0,\ y'(1)=-e$$

The exact solution of this problem $y(t) = t(1-t)e^{t}$.

Table (2) shows the comparison of the absolute error between the exact results and obtained approximate results by our method for m=3,K=2.

Table (2): Approximate results of example 3**.**

t	Exact solution	Approximate Solution	Absolute Error
0.0	0.0000	0.0000	1.6129e-021
0.1	0.0995	0.0995	1.9588e-005
0.2	0.1954	0.1954	1.3329e-006
0.3	0.2835	0.2836	1.2597e-004
0.4	0.3580	0.3580	2.1667e-006
0.5	0.4122	0.4118	3.3295e-004
0.6	0.4373	0.4375	2.3290e-004
0.7	0.4229	0.4225	4.4047e-004
0.8	0.3561	0.3565	4.4518e-004
0.9	0.2214	0.2205	8.5109e-004
1.0	0.0000	0.0000	5.1312e-014

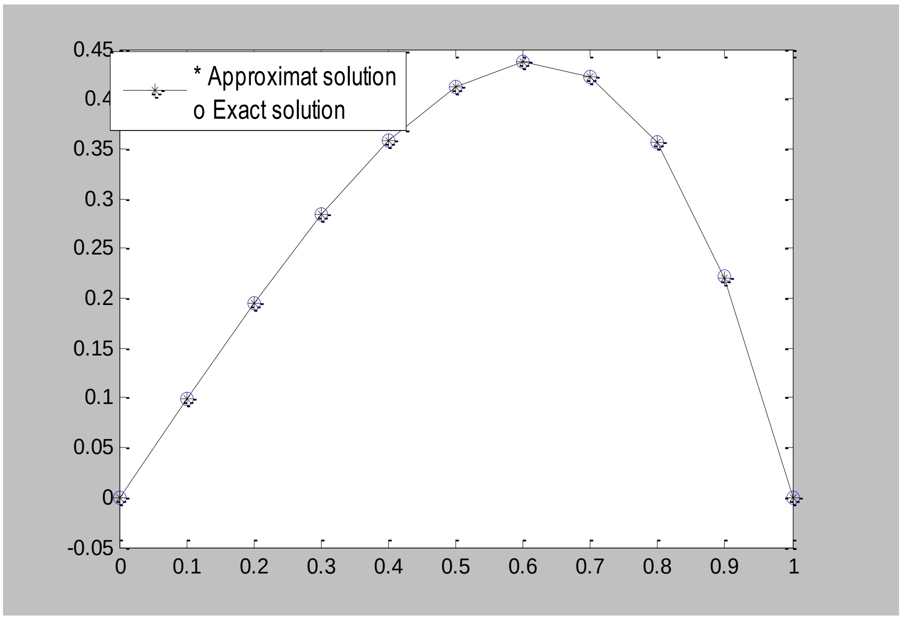

Figure (2): the plot of exact solution and approximate results using Chebyshev wavelets collocation when m=3,K=2 for example 3.

6. Conclusion

The aim of present work is to develop an efficient and accurate method for solving two points boundary value problem .The Chebyshev wavelet operational matrix of integral together with collocation method are used to reduce the problem into a system of linear equation in unknown approximate coefficients. Illustrative examples are included to demonstrate the validity and applicability of the technique.

7. References:

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تقريب موجي لحساب حل مسائل القيم الحدودية في نقطتين

إسراء هادي حسن
قسم العلوم التطبيقية
الجامعه التكنولوجية

الخلاصة:

في هذا البحث وضعت خوارزمية جديدة أساسها أستخدام مصفوفة العمليات التكاملية لدوال شبشف الموجية بدلا عن مصفوفة العمليات التفاضلية لأيجاد الحل التقريبي لمسائل القيم الحدوديه في نقطتين . توظيف متعددة حدود شبيشيف من النوع الاول بالاضافة الى طريقة التجميع لتحويل المعادلة التفاضليه مع شروطها الحدوديه الى نظام من المعادلات الجبريه الخطية .الدقة العالية للطريقه تظهر في الأمثلة المقترحة.