On Blow-up set of a Semilinear Heat Equation with Neumann Boundary Condition

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Abstract

This paper deals with the blow-up set of a semilinear heat equation defined on a ball, with nonlinear boundary condition, where the reaction term and the boundary term are powers of exponential types. In [9], under some restricted assumption, it has been proved that, in case of, the nonlinear terms are of exponential types without powers, the blow-up occurs only on the boundary. Our aim is to extend that result, for some regains of the powers those appear on the reaction and boundary terms.

1 Introduction

In this paper, we consider the initial –boundary problem:

$$\left\{\begin{array}{ll} u_t = \Delta u + \lambda e^{pu}, & (x,t) \in B_R \times (0,T), \\ \frac{\partial u}{\partial \mathbb{N}} = e^{qu}, & (x,t) \in \partial B_R \times (0,T), \\ u(x,0) = u_0(x), & x \in B_R, \end{array}\right\} \quad \ldots\ldots\ldots(1)$$

where $p > 0, q > 0$, $\lambda > 0$, B_R is a ball in R^n, $\mathbb{N}$ is the outward normal, u_0 is nonnegative symmetric, nondecreasing, smooth function satisfies the conditions

$$\frac{\partial u}{\partial \mathbb{N}} = e^{q u_0}, \qquad x \in \partial B_R \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$\Delta u_0 + \lambda e^{pu_0} \geq 0, \qquad u_{0r}(|x|) \geq 0, x \in \bar{B}_R, \ldots. \ldots. \ (3)$$

where $r = \sqrt{x_1^2 + x_2^2 + \cdots x_n^2}$.

It is known that, the existence and uniqueness of local classical solutions to this problem are guaranteed by the standard theory see [9],[5]. On the other hand, the nontrivial solutions of this problem blow-up in finite time and the blow-up set contains ∂B_R, and that due to comparison principle,[7], and the known blow-up results of problem where $\lambda = 0$ (see[2]).

In [9], it has been proved that, the lower blow-up rate is obtained as follows

$$\log c - \frac{1}{2\alpha}\log(T-t) \leq \max_{x \in \bar{B}_R} u(x,t) \quad \ldots\ldots.(4)$$

where $\alpha = \max\{p, q\}$,
while the upper blow-up rate estimate takes the from

$$\max_{x\in\bar{B}_R} u(x,t) \le \log C - \frac{1}{2q}\log(T-t) \quad \text{.......(5)}$$

Moreover, for the special case, where $p = q = 1$, and under some restricted assumptions on λ, it was shown that blow-up occurs only on the boundary.

In this paper, we shall extend some of these results showing that the blow-up occurs only on the boundary, where $1 < p \le 2, q \ge 1/2$, and for some values of λ.

2 Preliminaries

In the rest of paper, we denote for simplicity $u(x,t) = u(r,t)$.

The following lemma which has been proved in [9], shows some properties of the classical solutions to problem (1).

Lemma 2.1. Let u be a classical solution to problem (1), where u_0 satisfies the assumptions (2),(3). Then

1- $u > 0$, radial in $\bar{B}_R \times (0,T)$,
2- $u_r \ge 0$, in $[0,R] \times [0,T)$.
3- $u_t > 0$ in $\bar{B}_R \times (0,T)$.

3 Blow-up Set

In this section, we shall prove that, the blow-up of problem (1) occurs only on the boundary, with restricted assumptions on p, q and λ.

Theorem 3.1. *Suppose that u is a classical solution to following Problem:*

$$\left\{\begin{array}{ll} u_t = \Delta u + \lambda e^{pu}, & (x,t) \in B_R \times (0,T), \\ u(t,x) \le log\dfrac{C}{(T-t)^{\frac{1}{2q}}}, & (x,t) \in \partial B_R \times (0,T), \\ u(x,0) = u_0(x), & x \in B_R, \end{array}\right\} \quad \text{.........(6)}$$

where $n > 1, 1 < p \le 2, \; q \ge 1/2$.

Then for any $0 \le a \le R$, *and for some values of* λ such that

$$4\lambda R^2(n+1) \le \min\left\{\frac{1}{c}, \frac{4(n+1)}{R^2+4(n+1)T} e^{-\|u_0\|_\infty}\right\}, \quad \text{..........(7)}$$

There exists a positive constant A *such that,*

$$u(t,x) \le \log\left[\frac{1}{A(R^2-r^2)}\right] < \infty \quad \text{for } 0 \le |x| \le a < R, 0 < t < T.$$

Proof

Let $v(x) = A(R^2 - r^2)$, $r=|x|$, $0 \le r \le R$,
$z(x,t) = z(r,t) = log\frac{1}{[v(x)+B(T-t)]}$, *in* $\bar{B}_R \times (0,T)$,

where $B > 0, A \geq \lambda$.

A direct calculation shows that

$$z_t = \frac{B}{[v(x) + B(T-t)]},$$

$$z_r = \frac{4rA(R^2 - r^2)}{[v(x) + B(T-t)}$$

$$z_{rr} = \frac{[v(x)+B(T-t)][4A(R^2-3r^2)]+16A^2r^2(R^2-r^2)}{[v(x)+B(T-t)]^2}.$$

Thus

$$z_t - z_{rr} - \frac{n-1}{r} z_r - \lambda e^{pz} = \frac{[B - 4A(n-1)(R^2 - r^2)][v(x) + B(T-t)]}{[v(x) + B(T-t)]^2} - \frac{[4A(R^2 - 3r^2)][v(x) + B(T-t)] + 16r^2 v(x)}{[v(x) + B(T-t)]^2} - \frac{\lambda}{[v(x) + B(T-t)]^p}$$

$$\geq \frac{[B-4A(n-1)(R^2-r^2)-4A(R^2-3r^2)-16r^2]v(x)}{[v(x)+B(T-t)]^2} - \frac{\lambda}{[v(x)+B(T-t)]^2}$$

$$\geq \frac{[B - 4AR^2 n - 4AR^2]v(x) - \lambda}{[v(x) + B(T-t)]^2} \geq 0$$

Provided $(T-t) \leq 1/2$, $A(R^2 - r^2) \leq 1/2$, and

$$B \geq \frac{\lambda}{A(R^2-r^2)} + 4AR^2(n+1) \geq 4AR^2(n+1) \geq 4\lambda R^2(n+1), \quad \text{……..(8)}$$

where $0 < r \leq a < R$.

So, $\quad z_t - z_{rr} - \frac{n-1}{r} z_r - \lambda e^{pz} \geq 0$

Moreover,

$$z(x,0) = log \frac{1}{[v(x)+BT]} \geq log \frac{1}{[AR^4+BT]} \geq u(x,0), \; x \in B_R, \text{…(9)}$$

Provided $\frac{1}{[AR^4+BT]} \geq e^{u(x,0)}, \; x \in B_R,$

Or $\quad \frac{1}{[AR^4+BT]} \geq e^{||u_0||_\infty},$

From condition (8), we have

$$\frac{1}{[AR^4 + BT]} \geq \frac{1}{\frac{BR^2}{4(n+1)} + BT} = \frac{4(n+1)}{B[R^2 + 4(n+1)T]}$$

which leads to (9) is satisfied if

$$B \leq \frac{4(n+1)}{R^2 + 4(n+1)T} e^{-||u_0||_\infty}$$

and

$u(R,t) = log\frac{1}{B(T-t)} \geq log\frac{c}{(T-t)^{\frac{1}{2q}}} \geq u(R,t) \;\; t \in (0,T)$,….(10)

Provided $B \leq \frac{1}{c}$,

Thus, (9) and (10) can be satisfied when the following condition is held

$$B \leq \min\left\{\frac{1}{c}, \frac{4(n+1)}{R^2 + 4(n+1)T} e^{-\|u_0\|_\infty}\right\}$$

From above and comparison principle [7], we obtain

$$z(x,t) \geq u(x,t), \qquad \forall (x,t) \in B_R \times (0,T)$$

Thus

$$u(t,x) \leq \log[\frac{1}{A(R^2-r^2)}] < \infty \text{ for } 0 \leq |x| \leq a < R, 0 < t < T.$$

From above theorem and the upper blow-up rate estimate (5), which has been shown in [9], we conclude that, for $1 < p \leq 2,\; q \geq 1/2,$ and λ satisfies (7), the blow-up of problem (1) occurs only on the boundary.

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حول مجموعة التضخم لمعادلة الحرارة شبه الخطية مع شرط نيومان الحدودي

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الخلاصة

هذا البحث يتعامل مع مجموعة التضخم لمعادلة الحرارة شبة الخطية معرفة على كرة, مع شروط حدودية غير خطية, عندما مقطع رد الفعل والمقطع الحدودي هما قوى لدوال اسية. في المصدر 9 تحت شروط محددة تم برهان عندما الاجزاء غير الخطية التي تظهر في المعادلة دوال اسية بلا قوى, ان التضخم بالحل يظهر فقط على الحدود . هدفنا هو ان نعمم هذه النتيجة لبعض المديات من القوى والتي تظهر على مقطعي رد الفعل والحدود.