Efficient Modifications of the Adomian Decomposition Method for Thirteenth Order Ordinary Differential Equations

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Abstract

This paper deals with the thirteenth order differential equations linear and nonlinear in boundary value problems by using the Modified Adomian Decomposition Method (MADM), the analytical results of the equations have been obtained in terms of convergent series with easily computable components. Two numerical examples results show that this method is a promising and powerful tool for solving this problems.

Keywords : Modification adomian decomposition method; Boundary value problems; Linear and Nonlinear Problems; Approximate Solution.

1. Introduction

Over the last decade several analytical and approximate methods have been developed to solve the linear and nonlinear differential equations. Among them is the Adomian decomposition method. The Adomian decomposition method has been receiving much attention in recent years in applied mathematics in general, and in the area of series solutions in particular .The method proved to be powerful, effective, and can easily handle a wide class of linear or non-linear, ordinary or partial differential equations, and linear and non-linear integral equations differential delay. The method attacks the problem in a direct way and in a straightforward fashion without using linearization, perturbation or any other restrictive assumption that may change the physical behavior of the model under discussion. Many researchers use ADM to approximate numerical solutions. In [1], Wazwaz proposed a modification of ADM method in series solution to accelerate its rapid convergence, and, in [2] ,Wazwaz also presented several numerical examples of higher-order boundary value problems for first-order linear equation and second-order nonlinear equation by applying modified decomposition method. In addition, Wazwaz [3, 4] provided first-order linear and second-order nonlinear problems to solve fifth-order and sixth-order boundary value problems by the modified decomposition method. Later, Meˇstrovi´c [8], solved eight-order boundary value problems for first-order linear and second-order nonlinear boundary value problems. Similarly, in [9], Hosseini and Jafari used Adomian decomposition method to solve high-order and system of nonlinear differential equations., we use the same method for solving several different problems, such as, in calculus of variations, see [10], for eikonal partial differential equation, see [11], for the Fitzhugh-Nagumo equation which models the transmission of nerve impulses, see [12], for linear and nonlinear systems of Volterra functional equations using Adomian-Pade technique, see[13], for coupled Burgers equations by using Adomian-Pade technique see [14], for solution of a nonlinear time-delay model in biology by using semianalytical approaches, see[15], for solving the pantograph equation of order m, see[6], and for non classic problem for one-dimensional hyperbolic equation by using the decomposition procedure, see[16]. Although this paper is devoted to investigate ordinary differential equations, it seems useful to employ the Adomian decomposition method first to nonlinear ordinary differential equations. It is well known that nonlinear ordinary differential equations are, in general, difficult to handle .The Adomian decomposition method will be applied in a direct manner as discussed that non-linear terms should be represented by the so called adomian polynomials. It is interesting to point out that the modified decomposition method and the noise terms phenomenon, that will be used here at proper places. Recall that in solving differential equations, solutions are usually obtained as exact solutions defined in closed form expressions, or as series solutions normally obtained from concrete problems.

2. Modification Adomian Decomposition Method

To apply the Adomian decomposition method for solving ordinary differential equations, we consider the differential equation

$$L(y)+R(y)+N(y)=g(x), \qquad (2.1)$$

where the differential operator *L* is the highest order derivative in the equation, *R* is the remainder of the differential operator, where the order of *L* must be greater than *R*, N(y) expresses the nonlinear terms, and *g(x)* is an inhomogeneous term. Then, we assume that L is invertible by using the given conditions and applying the inverse operator L^{-1} to both sides of (2.1), we get the following equation:

$$y=\psi_0- L^{-1}g(x) - L^{-1}R(y) - L^{-1}N(y), \qquad (2.2)$$

where the function ψ_0 is arising from integrating the source term from applying the given conditions, which are prescribed. And so on the Adomian decomposition method admits the decomposition of y into an infinite series of components

$$y(x)=\sum_{n=0}^{\infty} y_n\,, \qquad (2.3)$$

and the nonlinear term N(y) be equated to an infinite series of polynomials

$$N(y)=\sum_{n=0}^{\infty} A_n\,, \qquad (2.4)$$

where A_n are the Adomian polynomials. Substituting (2.3) and (2.4) into (2.2) gives

$$\sum_{n=0}^{\infty} y_n=\psi_0- L^{-1}g(x) - L^{-1}R\left(\sum_{n=0}^{\infty} y_n\right) - L^{-1}N\left(\sum_{n=0}^{\infty} A_n\right), \qquad (2.5)$$

The various components y_n of the solution y can be easily determined by using the recursive relation

$$Y_0= \psi_0- L^{-1}g(x),$$
$$Y_{k+1}= -L^{-1}(Ry_k) - L^{-1}(Ny_k), \quad \text{for } k \geq 0. \qquad (2.6)$$

Consequently, the first few components can be written as

$$Y_0= \psi_0- L^{-1}g(x),$$
$$Y_1= -L^{-1}(Ry_0) - L^{-1}(A_0), \qquad (2.7)$$
$$Y_2= - L^{-1}(Ry_1) - L^{-1}(A_1),$$

Having determined the components Y_n , $n\geq 0$, the solution y in a series form follows immediately. As stated before, the series may be summed to provide the solution in a closed form. However, for concrete problems, the n-term partial sum

$$\varphi_n=\sum_{k=0}^{n-1} y_k\,, \qquad (2.8)$$

We can apply modification by assuming that the function f can be written as

$$f= \psi_0- L^{-1}g(x), \qquad (2.9)$$

The components Y_n are determined by using the following relation:

$$Y_0= f\,, \qquad (2.10)$$
$$Y_{k+1}= -L^{-1}(Ry_k) - L^{-1}(Ny_k), \quad \text{for } k \geq 0 \qquad (2.11)$$

From the above equations, we observe that the component Y_0 is identified by the

Function f. the modified Adomian decomposition method will minimize the volume of calculations, we split the function f into two parts, f_0 and f_1. Let the function be as follows:

$$f = f_0 + f_1\,, \qquad (2.12)$$

Under this assumption, we have a slight variation for components Y_0 and Y_1, where f_0

assigned to Y_0 and f_1 is combined with the other terms in (2.10) to assign Y_1. The modified recursive algorithm is as follows:

$$\left.\begin{aligned} y_0 &= f_0\,, \\ y_1 &= f_1 - L^{-1}(Ry_0) - L^{-1}(Ny_0), \\ y_{K+1} &= -L^{-1}(Ryk) - L^{-1}(Ny_K), \end{aligned}\right\} \qquad (2.13)$$

for $k \geq 1$.

However , the nonlinear term F(y) can be expressed by infinite series of the so-called Adomian polynomials A_n given in the form

$$F(y) = \sum_{n=0}^{\infty} A\,n\,(y_0, y_1, y_{2\,,\,\ldots\,,}\, y_n)\,, \qquad (2.14)$$

There are several rules that are needed to follow for Adomian polynomials of nonlinear operator F(y):

$$\begin{aligned} A_0 &= F(y_0)\,, \\ A_1 &= y_1\,F'(y_0)\,, \qquad (2.15) \\ A_2 &= y_2\,F'(y_0) + \tfrac{1}{2!}\,y^2{}_1\,F''(y_0)\,, \end{aligned}$$

and so on; see [7], then substituting (2.15) into (2.14) gives

$$F(y) = A_0 + A_1 + A_2 + \ldots \qquad (2.16)$$

To illustrate the applicability and effectiveness of the method, we presented two numerical examples, the results are compared with the methods DTM and VIM see [19,17]. The ADM [20,18,5] is a well-known systematic method for solving linear and nonlinear equations, including ordinary differential equations, partial differential equations, integral equations and integro-differential equations. The method permits us to solve both nonlinear initial value problems and boundary value problems. The method is well known, and several advanced progresses are conducted in this regard.

3. Numerical Examples

Example 1:-

The linear thirteenth order BVPs, is considered as

$$y^{(13)}(x) = Cosx - Sinx\,, \qquad 0 \leq x \leq 1\,, \qquad (3.1)$$

$$\left.\begin{aligned} &y(0) = 1\,, && y(1) = \cos 1 + \sin 1 \\ &y^{(1)}(0) = 1, && y^{(1)}(1) = \cos 1 - \sin 1 \\ &y^{(2)}(0) = -1\,, && y^{(2)}(1) = -\sin 1 - \cos 1 \\ &y^{(3)}(0) = -1\,, && y^{(3)}(1) = -\cos + \sin 1 \\ &y^{(4)}(0) = 1\,, && y^{(4)}(1) = \cos 1 + \sin 1 \\ &y^{(5)}(0) = 1, && y^{(5)}(1) = \cos 1 - \sin 1 \\ &y^{(6)}(0) = -1\,, && \end{aligned}\right\} \qquad (3.2)$$

The exact solution of the problem is $u(x) = Cosx + Sinx$.

Equation (3.1) can be rewritten in operator form as follows:

$$Ly = Cos\,x - Sin\,x\,, \qquad 0 \leq x \leq 1\,, \qquad (3.3)$$

Operating with thirteen fold integral operator L^{-1} on (3.3) and using the boundary conditions at x= 0 , we obtain the following equation:

$$y(x)=1+x-\frac{1}{2!}x^2-\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5-\frac{1}{6!}x^6+\frac{A}{7!}x^7+\frac{B}{8!}x^8+\frac{C}{9!}x^9+\frac{D}{10!}x^{10}+\frac{E}{11!}x^{11}+\frac{F}{12!}x^{12}+L^{-1}(Cosx-Sinx). \quad (3.4)$$

Then, determine the constants:

$y^{(7)}(0)=A, y^{(8)}(0)=B, y^{(9)}(0)=C, y^{(10)}(0)=D, y^{(11)}(0)=E, y^{(12)}(0)=F$.

Substituting the decomposition series (2.3) for y(x) into (3.4) yields

$$\sum_{n=0}^{\infty} y_n=1+x-\frac{1}{2!}x^2-\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5-\frac{1}{6!}x^6+\frac{A}{7!}x^7+\frac{B}{8!}x^8+\frac{C}{9!}x^9+\frac{D}{10!}x^{10}+\frac{E}{11!}x^{11}+\frac{F}{12!}x^{12}+L^{-1}(Cosx-Sinx). \quad (3.5)$$

Then, we split the terms into two parts which are assigned to $y_0(x)$ and $y_1(x)$ that are not included under $L-1$ in (3.5). We can obtain the following recursive relation:

$$y_0(x)=1,$$

$$y_1(x)=x-\frac{1}{2!}x^2-\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5-\frac{1}{6!}x^6+\frac{A}{7!}x^7+\frac{B}{8!}x^8+\frac{C}{9!}x^9+\frac{D}{10!}x^{10}+\frac{E}{11!}x^{11}+\frac{F}{12!}x^{12}+L^{-1}(Cosx-Sinx). \quad (3.6)$$

To determine the constants A,B,C,D,E and F, we use the boundary conditions in (3.2) at $x=1$ on the two-term approximant φ_2, where

$$\varphi_2=\sum_{k=0}^{1} y_k, \quad (3.7)$$

The coefficients A,B,C,D,E and F, were obtained by using Matlab with boundary conditions at $x=1$ in (3.2) given

A= -1.00000000000011, B= 1.00000000000391, C= 0.9999999999367303 ,

D= -0.9999999994259821, E= -1.000000002887327 , F= 1.000000006381563. (3.8)

Then we get the series solution as follows:

$$y(x)=\cos x+\sin x-2.179468943649792\times10^{-17}x^7+9.697411134499918\times10^{-17}x^8-1.743541143733518\times10^{-16}x^9+1.581839726078266\times10^{-16}x^{10}-7.233361753749345\times10^{-17}x^{11}+1.332263492302841\times10^{-17}x^{12}.$$

Example 2:-

The thirteenth order non-linear BVPs is considered as

$$y^{(13)}(x) = e^{-x}\, y^2(x)\,, \qquad 0 \le x \le 1\,, \tag{3.9}$$

$$\left.\begin{array}{ll} y(0) = 1, & y(1) = e \\ y^{(1)}(0) = 1, & y^{(1)}(1) = e \\ y^{(2)}(0) = 1, & y^{(2)}(1) = e \\ y^{(3)}(0) = 1, & y^{(3)}(1) = e \\ y^{(4)}(0) = 1, & y^{(4)}(1) = e \\ y^{(5)}(0) = 1, & y^{(5)}(1) = e \\ y^{(6)}(0) = 1, & \end{array}\right\} \tag{3.10}$$

The exact solution of the problem is $y(x) = e^x$.
Equation (3.9) can be rewritten in operator form as follows:

$$Ly = e^{-x}\, y^2(x)\,, \qquad 0 \le x \le 1\,, \tag{3.11}$$

Operating with thirteenfold integral operator L^{-1} on (3.11) and using the boundary conditions at x= 0 , we obtain the following equation:

$$Y(x) = 1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\frac{1}{6!}x^6+\frac{A}{7!}x^7+\frac{B}{8!}x^8+\frac{C}{9!}x^9+\frac{D}{10!}x^{10}+\frac{E}{11!}x^{11}+\frac{F}{12!}x^{12}+L^{-1}(e^{-x}\, y^2(x)). \tag{3.12}$$

Then, determine the constants:

$$y^{(7)}(0) = A,\, y^{(8)}(0) = B,\, y^{(9)}(0) = C,\, y^{(10)}(0) = D,\, y^{(11)}(0) = E,\, y^{(12)}(0) = F\,.$$

Substituting the decomposition series (2.3) for y(x) and the series of polynomials (2.4) into (3.12) yields

$$\sum_{n=0}^{\infty} y_n = 1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\frac{1}{6!}x^6+\frac{A}{7!}x^7+\frac{B}{8!}x^8+\frac{C}{9!}x^9+\frac{D}{10!}x^{10}+\frac{E}{11!}x^{11}+$$

$\frac{F}{12!}x^{12} \qquad\qquad +L^{-1}(e^{-x}\sum_{n=0}^{\infty}A_n)$.

(3.13)

Then, we split the terms into two parts, which are assigned to $y_0(x)$ and $y_1(x)$ that are not included under L^{-1} in (3.12). We can obtain the following recursive relation:

$$y_0(x) = 1 ,$$

$$y_1(x) = x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\frac{1}{6!}x^6+\frac{A}{7!}x^7+\frac{B}{8!}x^8+\frac{C}{9!}x^9+\frac{D}{10!}x^{10}+\frac{E}{11!}x^{11}+\frac{F}{12!}x^{12}+L^{-1}(e^{-x}A_0),$$

$$y_{k+1} = -L^{-1}(y_k), \qquad \text{for } k \geq 1. \tag{3.14}$$

To determine the constants A,B,C,D,E and F, we use the boundary conditions in (3.10) at $x=1$ on the two-term approximant φ_2, where

$$\varphi_2 = \sum_{k=0}^{1} y_k , \tag{3.15}$$

The coefficients A,B,C,D,E and F,are obtained by using Matlab which gives :

A= 0.9999992795332221, B= 1.000033584280779, C= 0.9992757215112231 ,

D=1.009033515968163, E=0.9339646023928783, F=1.236575305176785.
(3.16)

Then we get the series solution as follows:

$$y(x)=2+x^2-e^{-x}+0.08333333333333333x^4+0.002777777777777778x^6-1.429497575147045\times10^{-10}x^7+0.0000496040075466\,4629x^8-1.995917352229043\times10^{-9}x^9+5.536357793122142\times10^{-7}x^{10}-1.654325938129352\times10^{-9}x^{11}+4.669243913124268\times10^{-9}x^{12}.$$

The approximate solutions of two numerical examples obtained with the help of MADM are compared with the results of the VIM and DTM , in Tables 1-2 respectively. From the numerical results, it is clear that the MADM is more efficient and accurate. The graphical comparison of exact and approximate solutions is shown in Figure 1-2 respectively.

4. Conclusions

In this paper, modified adomian decomposition method can be used successfully for finding the solution of linear and nonlinear BVPs of thirteenth order ODEs. It may be concluded that is a very powerful and efficient in finding highly accurate solutions for a large class of differential equations and is practically well suited for nonlinear problems.

Table 1: Comparison of numerical results for Example 1

x	Exact solution	MADM	Error MADM	Error DTM	Error VIM
0	1.000000000000000	1.000000000000000	0	0.0000	0
0.1	1.094837581924854	1.094837581924854	0	2.22045E-16	3.88578E-15
0.2	1.178735908636303	1.178735908636303	0	0.0000	1.46216E-13
0.3	1.250856695786946	1.250856695786946	0	2.22045E-15	8.80518E-13
0.4	1.310479336311536	1.310479336311536	0	6.66134E-15	2.35822E-12
0.5	1.357008100494576	1.357008100494576	0	1.11022E-14	3.8014E-12
0.6	1.389978088304714	1.389978088304714	0	1.04361E-14	5.14766E-11
0.7	1.409059874522180	1.409059874522180	0	5.32907E-15	1.56224E-11
0.8	1.414062800246688	1.414062800246688	0	8.88178E-16	8.99409E-11
0.9	1.404936877898148	1.404936877898148	0	0.0000	4.70031E-10
1	1.381773290676036	1.381773290676036	0	0.0000	2.06386E-9

Table 2: Comparison of numerical results for Example 2

x	Exact solution	MADM	Error MADM	Error DTM	Error VIM
0	1.000000000000000	1.000000000000000	0	0.0000	0
0.1	1.105170918075648	1.105170918075648	0	4.44089E-16	4.17444E-14
0.2	1.221402758160170	1.221402758160170	0	4.44089E-16	2.64144E-12
0.3	1.349858807576003	1.349858807576004	4.E-15	2.44249E-15	2.99314-11
0.4	1.491824697641270	1.491824697641273	1.2E-14	7.32747E-15	1.67101E-10
0.5	1.648721270700128	1.648721270700133	1.9E-14	1.22125E-14	6.30955E-10
0.6	1.822118800390509	1.822118800390514	1.8E-14	1.11022E-14	1.84757E-09
0.7	2.013752707470477	2.013752707470479	1.E-14	5.77316E-15	4.47866E-09
0.8	2.225540928492468	2.225540928492469	2.E-15	1.77636E-15	9.21592E-09
0.9	2.459603111156950	2.459603111156950	0	8.88178E-16	1.58906E-08
1	2.718281828459046	2.718281828459046	0	0.0000	2.09057E-08

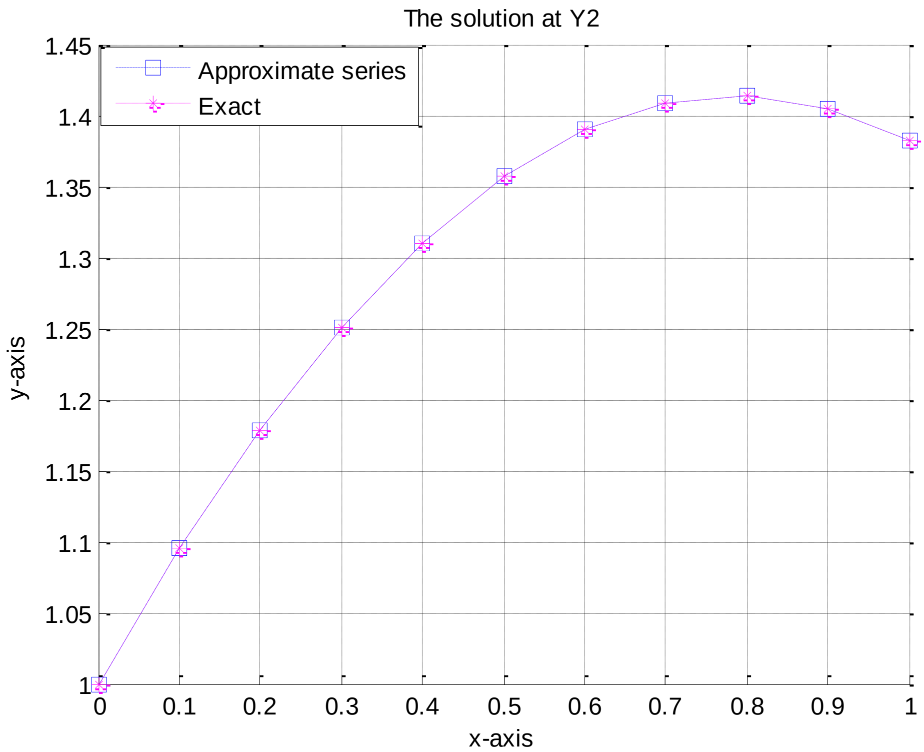

Figure 1: Comparison between the exact and MADM solution of example1.

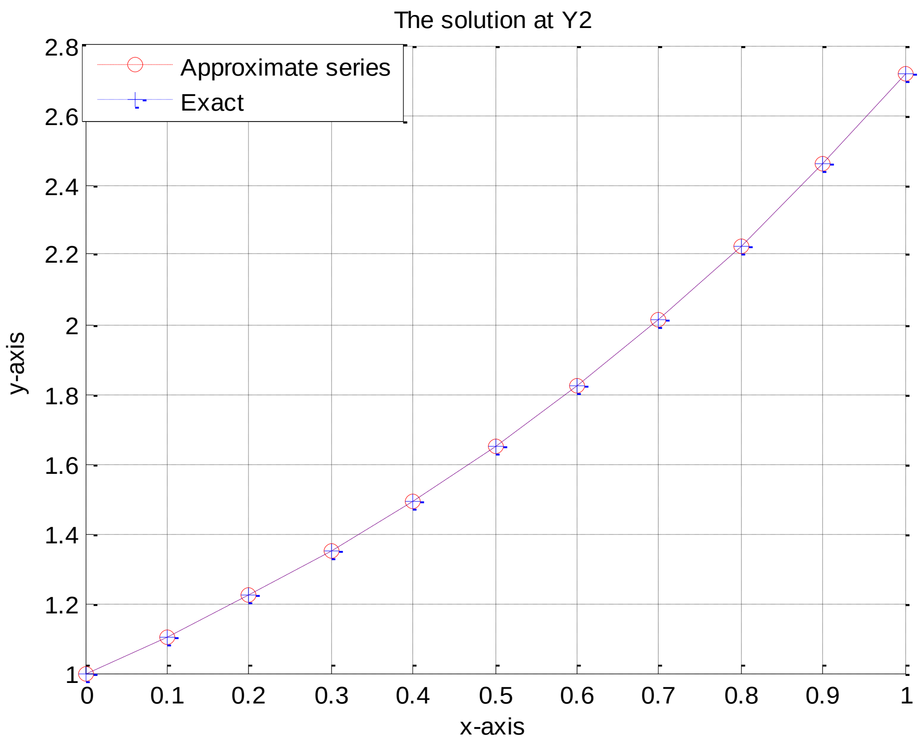

Figure 2: Comparison between the exact and MADM solution of example2.

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كفاءة طريقة أدومين التحليلة المطورة لحل معادلات تفاضلية اعتيادية من الرتبة الثالثة عشر

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خلاصة

هذا البحث يعرض طريقة لحل معادلات تفاضلية اعتيادية خطية و غير خطية من الرتبة الثالثة عشر ذات الشروط الحدودية بأستخدام طريقة أدومين التحليلة المطورة, النتائج التحليلية لهذه المعادلات حصلنا عليها بسهولة من المتسلسلة المتقارية , ونتائج الأمثلة أثبتت دقة وكفاءة الطريقة وسهولة الأداء لحل هذه المسائل .