Scraton Method for Solving nth Order State-Space Equations of Linear Continuous-Time Control Systems

Raghad Kadhim Salih , **Atheer Jawad Kadhim**
Department of Applied Sciences
University of Technology
Department of Applied Sciences
University of Technology
&
Isra'a Hadi Hasan
Department of Applied Sciences
University of Technology

Abstract

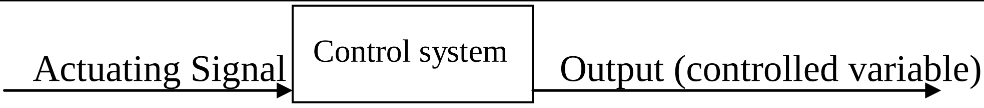
The paper presents a developed method with algorithms written in Matlab language to find the numerical solution of nth-order state-space equations (SSE) of linear continuous-time control system by using Scraton method. State-space equation is the modern representation to analysis continuous-time system. It was treated numerically to the single-input-single-output (SISO) systems as well as multiple-input-multiple-output (MIMO) systems by using fourth-order Scraton method. Scraton method employed to find the output values of the state-space equations. Comparisons between the numerical and exact results are given for some numerical examples to conciliate the accuracy of the proposed method.

Key words: State-space equation, Scraton method, Control system and Algorithms.

1. Introduction

Control systems are playing vital role in our life for instance: thermostat, automatic control of airplane, etc. The system is a combination of component that act together and perform a certain objective [1,2].

In recent years, automatic control systems have assumed an increasingly important role in the development and advancement of modern civilization and technology. They are employed in numerous applications, such as quality control of manufactured products and machine tooling. The basic control system problem may be described by the simple block diagram shown in figure (1) [1,3,4].



Figure(1) The basic control system.

Modern control theory adopts what known as state-space equations (SSE) for mathematical representation of systems. Among its different advantages it makes possible to deal with [4,5]:

- Time variant systems.
- Nonlinear systems.
- Multiple-input-multiple-output system.

The linear state-space equation is given by:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

where $x(t) \in lR^n$ is the state vector, $u(t) \in lR^m$ is the control input of the system, $y(t) \in lR^p$ is the output of the system, A is the system matrix, B is the control input matrix, C is the output or measurement matrix and D is the direct feed matrix. This description is said to be time-invariant if A,B,C and D are constant matrices.

State-space method of continuous-time system was solved by several methods as Laplace transformation and matrix exponential [6,7]. In this work different types of state-space representation are solved numerically by using fourth order Scraton method. The modeling of linear continuous-time systems by using state space method with their solutions have been presented in the following section.

2. State - Space Equations :

State space equation (SSE) describes the state of the system, where it is ideally suited for the analysis of multiple-input-multiple-output systems as well as single-input-single-output systems [8].

J. John [9] used state-space representation for solving the pitch controller problem and Dk. James [10] used state space equations to solve the cruise control problem.

In this section, we shall present methods for obtaining state-space representation (SSR) of linear continuous-time control systems (CTCS).

2.1 SSR of n^{th} Order Linear CTCS In Which The Forcing Function Dose Not Involve Derivative Terms :

Consider the following nth-order dynamic system [1,4]:

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + ... + a_{n-1}\dot{y}(t) + a_n y(t) = u(t) \qquad \ldots(1)$$

Noting that the knowledge of $y(0), \dot{y}(0),..., y^{(n-1)}(0)$, together with the input or forcing function u(t) for $t \geq 0$, determines completely the future behavior of the system, we may take $y(t), \dot{y}(t),..., y^{(n-1)}(t)$ as a set of n state variables.

Let us define the following state equation:

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad \text{…(2)}$$

where x (t) is the state vector which is:

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ \dot{y}(t) \\ \vdots \\ y^{(n-1)}(t) \end{bmatrix} \qquad \text{…(3)}$$

A is the time-invariant system matrix, defined by:

$$A = \begin{bmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & ... & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & ... & -a_1 \end{bmatrix}_{n \times n}$$

and B is the $(n \times 1)$ time-invariant input matrix defined by : $B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{n \times 1}$

The output equation becomes : $y(t) = [1 \quad 0 \quad \cdots \quad 0] \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ …(4)

or $\qquad y(t) = Cx(t) \qquad$ … (5)

where $\qquad C = [1 \quad 0 \quad \cdots \quad 0]$.

Hence, Eq.(2) and Eq.(5) are the state-space equation.

2.2 SSR of n^th Order Linear CTCS with (m) Forcing Functions [2,6,7]:

Consider the multiple-input-multiple-output (MIMO) linear continuous system shown in figure (2). $x_1(t), x_2(t),..., x_n(t)$ represent the state variables, $u_1(t), u_2(t),..., u_m(t)$ denote the input variables and $y_1(t), y_2(t),..., y_p(t)$ are the output variables. From figure(2) we obtain the system equations as follows:

$$\left.\begin{aligned}
\dot{x}_1(t) &= a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t) + b_{11}u_1(t) + b_{12}u_2(t) + \cdots + b_{1m}u_m(t) \\
\dot{x}_2(t) &= a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t) + b_{21}u_1(t) + b_{22}u_2(t) + \cdots + b_{2m}u_m(t) \\
&\vdots \\
\dot{x}_n(t) &= a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t) + b_{n1}u_1(t) + b_{n2}u_2(t) + \cdots + b_{nm}u_m(t)
\end{aligned}\right\} \quad ...(6)$$

where the a's and b's are constants. Hence, the state equation for the system (6) is:

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad \ldots(7)$$

where x (t) is the state vector given by :

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} ,$$

while the input vector u(t) is given by :

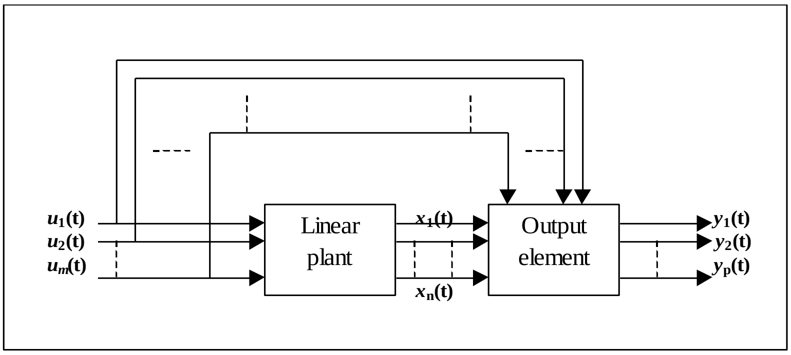
$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix} .$$

A is the time-invariant system matrix defined by :

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}_{n\times n}$$

and B is the time-invariant input matrix defined by :

$$B = \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1m} \\ b_{21} & b_{22} & \ldots & b_{2m} \\ \vdots & & & \\ b_{n1} & b_{n2} & \ldots & b_{nm} \end{bmatrix}_{n\times m}$$



Figure(2) (Multiple-input-multiple-output linear continuous system)

Similarly, the continuous time output variables are linear combinations of the values of the input and state variables as follows:

$$\left.\begin{aligned} y_1(t) &= c_{11}x_1(t) + c_{12}x_2(t) + \cdots + c_{1n}x_n(t) + d_{11}u_1(t) + d_{12}u_2(t) + \cdots + d_{1m}u_m(t) \\ y_2(t) &= c_{21}x_1(t) + c_{22}x_2(t) + \cdots + c_{2n}x_n(t) + d_{21}u_1(t) + d_{22}u_2(t) + \cdots + d_{2m}u_m(t) \\ &\vdots \\ y_p(t) &= c_{p1}x_1(t) + c_{p2}x_2(t) + \cdots + c_{pn}x_n(t) + d_{p1}u_1(t) + d_{p2}u_2(t) + \cdots + d_{pm}u_m(t) \end{aligned}\right\} \quad ...(8)$$

System (8) can be written in matrix form as:

$$y(t) = Cx(t) + Du(t) \qquad \ldots(9)$$

where the output vector y(t) is given by: $y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix}$,

C is the $p \times n$ time-invariant output matrix defined by :

$$C = \begin{bmatrix} c_{11} & c_{12} & \ldots & c_{1n} \\ c_{21} & c_{22} & \ldots & c_{2n} \\ \vdots & & & \\ c_{p1} & c_{p2} & \ldots & c_{pn} \end{bmatrix}_{p \times n} ,$$

and D is the $p \times m$ time-invariant transmission matrix defined by :

$$D = \begin{bmatrix} d_{11} & d_{12} & \ldots & d_{1m} \\ d_{21} & d_{22} & \ldots & d_{2m} \\ \vdots & & & \\ d_{p1} & d_{p2} & \ldots & d_{pm} \end{bmatrix}_{p \times m} .$$

Eq.(9) is the output equation for the system. The matrices A, B, C and D completely characterize the system dynamics.

Eq.(7) and Eq.(9) are the state-space equation of the continuous system. Note that, when the technique:

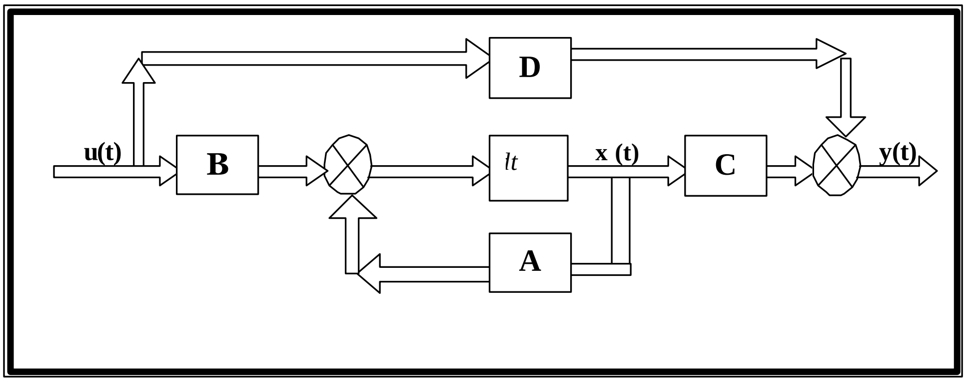
$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}\right\}$$

…(10)

1- Has one input (m = 1) and one output (p = 1), then the system is called system with single-input-single-output (SISO).
2- Has one input (m = 1) and (p) outputs, then the system is called system with single-input-multiple-output (SIMO).
3- Has (m) inputs and one output (p = 1), then the system is called system with multiple-input-single-output (MISO).

4- Has (m) inputs and (p) outputs, then the system is called system with multiple-input-multiple-output (MIMO).

A block diagram representation of the system defined by Eq.(10) is shown in figure (3). Double lines are used in the diagram to indicate vector quantities [1,4].



Figure(3) (Block diagram of the continuous-time system described by state-space technique in Eq.(7) and Eq.(9)).

2.3 SSR of n^th Order Linear CTCS in which The Forcing Function Involves Derivative Terms[1,2,5] :

If the system involves derivatives of the forcing function, such as:

$$y^{(n)}(t)+a_1y^{(n-1)}(t)+\dots+a_{n-1}\dot{y}(t)+a_ny(t)=b_0u^{(n)}(t)+b_1u^{(n-1)}(t)+\cdots+b_{n-1}\dot{u}(t)+b_nu(t) \quad \dots(11)$$

then, we define the following n variables as a set of n state variables:

$$\left.\begin{aligned}
x_1(t)&=y(t)-\beta_0u(t)\\
x_2(t)&=\dot{y}(t)-\beta_0\dot{u}(t)-\beta_1u(t)=\dot{x}_1(t)-\beta_1u(t)\\
x_3(t)&=\ddot{y}(t)-\beta_0\ddot{u}(t)-\beta_1\dot{u}(t)-\beta_2u(t)=\dot{x}_2(t)-\beta_2u(t)\\
&\vdots\\
x_n(t)&=\overset{(n-1)}{y}(t)-\beta_0\overset{(n-1)}{u}(t)-\beta_1\overset{(n-2)}{u}(t)-\cdots-\beta_{n-2}\dot{u}(t)-\beta_{n-1}u(t)=\dot{x}_{n-1}(t)-\beta_{n-1}u(t)
\end{aligned}\right\} \quad \dots(12)$$

where $\beta_0,\beta_1,\beta_2,\dots,\beta_n$ are determined from

$$\left.\begin{aligned}
\beta_0&=b_0\\
\beta_1&=b_1-a_1\beta_0\\
\beta_2&=b_2-a_1\beta_1-a_2\beta_0\\
\beta_3&=b_3-a_1\beta_2-a_2\beta_1-a_3\beta_0\\
&\vdots\\
\beta_n&=b_n-a_1\beta_{n-1}-\cdots-a_{n-1}\beta_1-a_n\beta_0
\end{aligned}\right\} \quad \dots(13)$$

Hence, the state equation and the output equation of state-space method are:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_{n-1}(t) \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{n-1} \\ \beta_n \end{bmatrix} [u(t)]$$

$$y(t) = [1 \quad 0 \quad 0 \quad \cdots \quad 0] \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} + \beta_0 u(t)$$

or

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad ...(14)$$

$$y(t) = Cx(t) + Du(t) \qquad ...(15)$$

where

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} , \quad A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} , \quad B = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{n-1} \\ \beta_n \end{bmatrix}$$

$$, \quad C = [1 \quad 0 \quad 0 \quad \cdots \quad 0] \quad and \quad D = \beta_0 = b_0 \quad .$$

The initial condition x(0) may be determined by using Eq.(12).

3. Scraton Method:

Scraton's method provides efficient mean for the solution of the many problem arising in various fields of science and engineering [11,12].

Let $y' = f(t, y(t))$ with $y(t_0) = y_0$ ... (16)

Hence, a fourth order method is derived by Scraton [9,10] as:

$$\left.\begin{aligned} y_{n+1} &= y_n + h\left[\tfrac{17}{162}S_1 + \tfrac{81}{170}S_3 + \tfrac{32}{135}S_4 + \tfrac{250}{1377}S_5\right] \\ &\textit{where} \\ S_1 &= f(t_n, y_n), \\ S_2 &= f(t_n + \tfrac{2}{9}h, y_n + \tfrac{2}{9}hS_1), \\ S_3 &= f(t_n + \tfrac{1}{3}h, y_n + \tfrac{1}{12}hS_1 + \tfrac{1}{4}hS_2), \\ S_4 &= f\left(t_n + \frac{3}{4}h, y_n + \frac{3h}{128}(23S_1 - 81S_2 + 90S_3)\right), \\ S_5 &= f\left(t_n + \frac{9}{10}h, y_n + \frac{9h}{10000}(-345S_1 + 2025S_2 - 1224S_3 + 244S_4)\right). \end{aligned}\right\} \quad \ldots(17)$$

The local truncation error (L.T.E.) of eq.(17) is given by

$$E_{n+1} = hqr/g \quad \ldots (18)$$

where

$$\left.\begin{aligned} q &= -\tfrac{1}{18}S_1 + \tfrac{27}{170}S_3 - \tfrac{4}{15}S_4 + \tfrac{25}{153}S_5 \ , \\ r &= \tfrac{19}{24}S_1 - \tfrac{27}{8}S_2 + \tfrac{57}{20}S_3 - \tfrac{4}{15}S_4 \ , \\ g &= S_4 - S_1. \end{aligned}\right\} \quad \ldots (19)$$

4. Numerical Solution of State-Space Equations (SSE) of Linear Continuous-Time Systems Using Scraton Method :

In this section different types of linear state-space equations have been solved using Scraton method.

4.1 The Solution of n^th Order SSE In Which The Forcing Function Dose Not Involve Derivative Terms :

Recall eq.(2) , it can be written as:

$$\frac{dx_i(t)}{dt} = f_i(x_1(t), x_2(t), \ldots, x_n(t), u(t)) \quad \ldots(20)$$

where $x_1(t), x_2(t), \ldots, x_n(t)$ are the state variables, u(t) is the input of the system and f_i , i=1,2,…,n denotes the ith linear functional relationship.

The output of the system is obtained from eq.(5) as:

$$y(t) = x_1(t) \quad \ldots(21)$$

The numerical solution of SSE in eq.(20) and eq.(21) can be found by using (SS-SSES) algorithm which summarizes the steps for finding the numerical solution for the **SISO-SSE** using **Scraton** method in eq.(17).

SS-SSES Algorithm :

Input

- t_0 (the initial state).
- k ((k + 1) is the number of points $(t_0, t_1, \ldots, t_k)$).

- n (the order of the SSE).
- The functions f_i , i=1,2,…,n of state equation (20).

Output

- The state variables $x_i(t_j)$ where i=1,2,…,n and j=0,1,…, k.
- The output values of SSE $y(t_j)$, j=0,1,…, k.

Step 1: Set $h=\frac{t_k-t_0}{k}$

Step 2: Set j=0

Step 3: For each i=1,2,…,n compute S_{1i} where $S_{1i}=f_i(x_1(t_j),\dots,x_n(t_j),u(t_j))$

Step 4: $\forall\, i=1,2,\dots,n$ compute S_{2i} where

$$S_{2i}=f_i\left(x_1(t_j)+\tfrac{2}{9}hS_{11},\dots,x_n(t_j)+\tfrac{2}{9}hS_{1n},u(t_j+\tfrac{2}{9}h)\right)$$

Step 5: $\forall\, i=1,2,\dots,n$ compute S_{3i} where

$$S_{3i}=f_i\left(x_1(t_j)+\tfrac{1}{12}hS_{11}+\tfrac{1}{4}hS_{21},\dots,x_n(t_j)+\tfrac{1}{12}hS_{1n}+\tfrac{1}{4}hS_{2n},u(t_j+\tfrac{1}{3}h)\right).$$

Step 6: $\forall\, i=1,2,\dots,n$ compute S_{4i} where

$$S_{4i}=f_i\left(\begin{array}{l}x_1(t_j)+\dfrac{3h}{128}(23S_{11}-81S_{21}+90S_{31}),\dots,x_n(t_j)+\dfrac{3h}{128}(23S_{1n}-\\ 81S_{2n}+90S_{3n}),u(t_j+\tfrac{3}{4}h)\end{array}\right)$$

Step 7: $\forall\, i=1,2,\dots,n$ compute S_{5i} where

$$S_{5i}=f_i\left(\begin{array}{l}x_1(t_j)+\dfrac{9h}{10000}(-345S_{11}+2025S_{21}-1224S_{31}+244S_{41}),\dots,x_n(t_j)\\ +\dfrac{9h}{10000}(-345S_{1n}+2025S_{2n}-1224S_{3n}+244S_{4n}),u(t_j+\tfrac{9}{10}h)\end{array}\right).$$

Step 8: $\forall\, i=1,2,\dots,n$ compute :

$$t_{j+1}=t_j+h$$

$$x_i(t_{j+1})=x_i(t_j)+h\left[\tfrac{17}{162}S_{1i}+\tfrac{81}{170}S_{3i}+\tfrac{32}{135}S_{4i}+\tfrac{250}{1377}S_{5i}\right]$$

Step 9: Put j = j+1

Step 10: If j = k then go to (step 11).
Else go to (step 3)

Step 11: For j=0,1,…, k compute the output values of SSE :

$$y(t_j)=x_1(t_j)\quad .$$

4.2 The Solution of n[th] Order SSE with (m) Forcing Functions

The fourth order Scraton method has been used to find the numerical solution for the following MIMO-SSE:

eq.(6) in section (2.2) can be written as:

$$\frac{dx_i(t)}{dt} = f_i(x_1(t), x_2(t),..., x_n(t), u_1(t), u_2(t),..., u_m(t)) \qquad \ldots(22)$$

where $x_1(t), x_2(t),..., x_n(t)$ are the state variables, $u_1(t), u_2(t),..., u_m(t)$ are the input variables of the system and f_i , i=1,2,…,n denotes the ith linear functional relationship.

The outputs ($y_q(t)$, q =1,2,…,p) of the system in eq.(8) are related to the state variables and the input through the following expression:

$$y_q(t) = g_q(x_1(t), x_2(t),..., x_n(t), u_1(t), u_2(t),..., u_m(t)) \qquad \ldots(23)$$

where g_q , q=1,2,…,p denotes the qth linear functional relationship.

The numerical solution of SSE in eq.(22) and eq.(23) can be found using (MM-SSES) algorithm where it summarizes the steps for finding the numerical solution for the **MIMO-SSE** using **Scraton** method in eq.(17).

MM-SSES Algorithm:

Input

- t_0 (the initial state).
- k where (k + 1) is the number of points ($t_0, t_1,..., t_k$).
- n (the order of the SSE).
- The functions f_i , i=1,2,…,n of state equation (25).
- The functions g_q , q=1,2,…,p of the output equation (26) of state space model.

Output

- The state variables $x_i(t_j)$ where i=1,2,…,n and j=0,1,…, k.
- The output values of SSE $y_q(t_j)$ in eq.(23) where $q = 1,2,..., p$ and j=0,1,…, k.

Step 1: Set $h = \frac{t_k - t_0}{k}$.

Step 2: Set j=0

Step 3: For each i=1,2,…,n compute S_{1i} where

$$S_{1i} = f_i(x_1(t_j),..., x_n(t_j), u_1(t_j),..., u_m(t_j)).$$

Step 4: $\forall\, i = 1,2,..., n$ compute S_{2i} where

$$S_{2i} = f_i\left(x_1(t_j) + \tfrac{2}{9} hS_{11},..., x_n(t_j) + \tfrac{2}{9} hS_{1n}, u_1(t_j + \tfrac{2}{9} h),..., u_m(t_j + \tfrac{2}{9} h)\right).$$

Step 5: $\forall\, i = 1,2,..., n$ compute S_{3i} where

$$S_{3i} = f_i\left(x_1(t_j) + \tfrac{1}{12} hS_{11} + \tfrac{1}{4} hS_{21},..., x_n(t_j) + \tfrac{1}{12} hS_{1n} + \tfrac{1}{4} hS_{2n}, u_1(t_j + \tfrac{1}{3} h),..., u_m(t_j + \tfrac{1}{3} h)\right).$$

Step 6: $\forall\, i = 1,2,..., n$ compute S_{4i} where

$$S_{4i} = f_i\begin{pmatrix} x_1(t_j) + \frac{3h}{128}(23S_{11} - 81S_{21} + 90S_{31}),..., x_n(t_j) + \frac{3h}{128}(23S_{1n} - \\ 81S_{2n} + 90S_{3n}), u_1(t_j + \frac{3}{4} h),..., u_m(t_j + \frac{3}{4} h) \end{pmatrix}.$$

Step 7: $\forall\, i = 1,2,...,n$ compute S_{5i} where

$$S_{5i} = f_i\left(\begin{matrix} x_1(t_j) + \dfrac{9h}{10000}(-345S_{11} + 2025S_{21} - 1224S_{31} + 244S_{41}),...,x_n(t_j) \\ + \dfrac{9h}{10000}(-345S_{1n} + 2025S_{2n} - 1224S_{3n} + 244S_{4n}), u_1(t_j + \frac{9}{10}h),...,u_m(t_j + \frac{9}{10}h) \end{matrix}\right).$$

Step 8: $\forall\, i = 1,2,...,n$ compute :

$$t_{j+1} = t_j + h$$

$$x_i(t_{j+1}) = x_i(t_j) + h\left[\tfrac{17}{162}S_{1i} + \tfrac{81}{170}S_{3i} + \tfrac{32}{135}S_{4i} + \tfrac{250}{1377}S_{5i}\right]$$

Step 9: $\forall\, q = 1,2,..., p$ compute the output values of MIMO-SSE :

$$y_q(t_j) = g_q(x_1(t_j), x_2(t_j),..., x_n(t_j), u_1(t_j), u_2(t_j),..., u_m(t_j)) .$$

Step 10: Put j = j+1

Step 11: If j = k then stop.
Else go to (step 3)

4.3 The Solution of n^{th} Order SSE In Which The Forcing Function Involves Derivative Terms :

The fourth order Scraton method has been used to find the numerical solution for the following SSE:

eq.(14) in section (2.3) can be written as:

$$\frac{dx_i(t)}{dt} = f_i(x_1(t), x_2(t),..., x_n(t), \beta_i\, u(t)) \qquad \text{… (24)}$$

where $x_1(t), x_2(t),..., x_n(t)$ are the state variables, $u(t)$ are the input variable of the system, β_i in eq.(13) and f_i , i=1,2,…,n denotes the ith linear functional relationship.

The output of the system is obtained from eq.(15) as:

$$y(t) = x_1(t) + \beta_0\, u(t) \qquad \text{… (25)}$$

The numerical solution of SSE in eq.(24) and eq.(25) can be found using fourth order Scraton method by applying (SS-SSES) algorithm as prescribed in section (4.1).

5. Numerical Examples:

The previous methods in section (4) are illustrated in the following examples :-

Example (1) :

In the Cruise Control Problem [10], the state-space model was derived as:

$$\begin{bmatrix}\dot{x}(t)\\ \ddot{x}(t)\end{bmatrix}=\begin{bmatrix}0 & 1\\ 0 & -0.05\end{bmatrix}\begin{bmatrix}x(t)\\ \dot{x}(t)\end{bmatrix}+\begin{bmatrix}0\\ 0.001\end{bmatrix}u(t)$$

$$y(t)=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x(t)\\ \dot{x}(t)\end{bmatrix}+\begin{bmatrix}0\end{bmatrix}u(t)$$

where the initial state is: $x(0)=\begin{bmatrix}0\\ 1\end{bmatrix}$ and the forcing function $u(t)=e^{t}$, $t\geq 0$.

The exact solution of the above SISO state-space model is:

$$x(t)=\begin{bmatrix}exact_1\\ exact_2\end{bmatrix}=\begin{bmatrix}x(t)\\ \dot{x}(t)\end{bmatrix}=\begin{bmatrix}\frac{-2098}{105}e^{-\frac{1}{20}t}+\frac{999}{50}+\frac{1}{1050}e^{t}\\ \frac{1049}{1050}e^{-\frac{1}{20}t}+\frac{1}{1050}e^{t}\end{bmatrix}$$

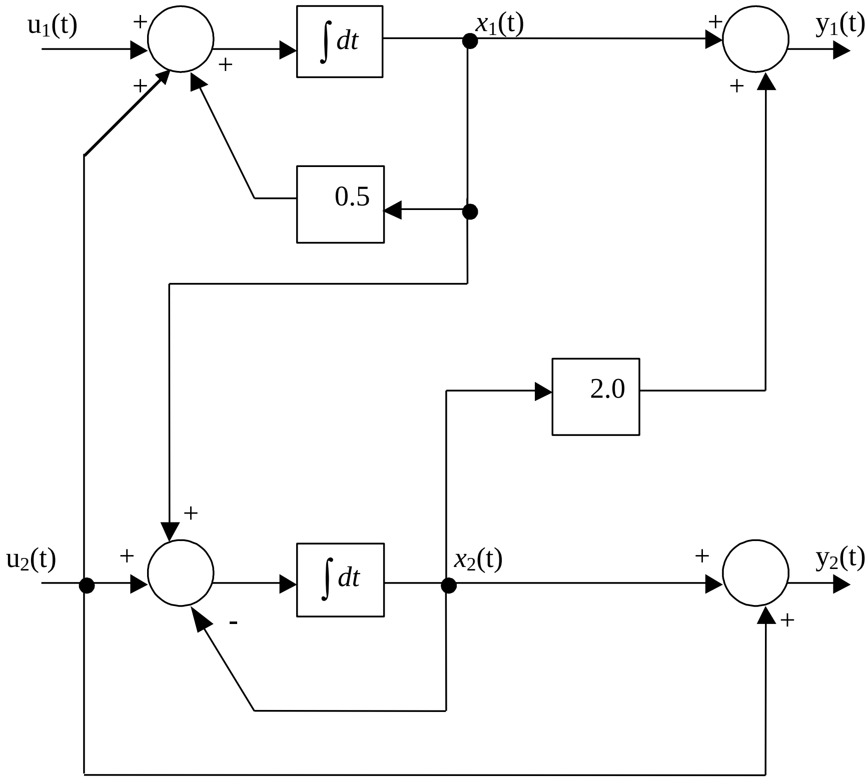
Table (1) presents the comparison between the exact and numerical solution by applying (SS-SSES) algorithm for k=10, h=0.1 and $t_i = ih$, $i=0,1,\ldots,k$ depending on least square error (L.S.E.). The local truncation error (L.T.E.) in eq.(18) and the output variables y(t) of state space model by applying (SS-SSES) algorithm is also tabulated.

Table (1) The numerical solution x(t) and the output variables y(t) of state space model for Ex.(1).

t	Exact$_1$	Scraton $x_1(t)$	L.T.E.	Exact$_2$	Scraton $x_2(t)$	L.T.E.	output y(t)	Scraton y(t)
0	0.0000	0.0000	0.0000	1.0000	1.0000	0.0000	0.0000	0.0000
0.1	0.0998	0.0998	-4.4e-14	0.9951	0.9951	-1.1e-14	0.0998	0.0998
0.2	0.1990	0.1990	-4.8e-14	0.9903	0.9903	-1.2e-14	0.1990	0.1990
0.3	0.2978	0.2978	-5.3e-14	0.9855	0.9855	-1.4e-14	0.2978	0.2978
0.4	0.3961	0.3961	-5.8e-14	0.9807	0.9807	-1.5e-14	0.3961	0.3961
0.5	0.4939	0.4939	-6.5e-14	0.9760	0.9760	-1.7e-14	0.4939	0.4939
0.6	0.5913	0.5913	-7.2e-14	0.9713	0.9713	-1.9e-14	0.5913	0.5913
0.7	0.6882	0.6882	-8.1e-14	0.9666	0.9666	-2.1e-14	0.6882	0.6882
0.8	0.7846	0.7846	-9.2e-14	0.9620	0.9620	-2.4e-14	0.7846	0.7846
0.9	0.8806	0.8806	-10.4e-13	0.9574	0.9574	-2.6e-14	0.8806	0.8806
1	0.9761	0.9761	-11.9e-13	0.9529	0.9529	-2.9e-14	0.9761	0.9761
L.S.E.		**3.5652e-21**		**L.S.E.**	**7.1377e-24**	**-**	**L.S.E.**	**3.5652e-21**

Example (2) :

Consider the MIMO control system shown in figure (4) :



Figure(4) (Simulation diagram for a multivariable system)

The MIMO state-space model was derived from fig.(4) as follows :

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

The initial state of the MIMO state-space model is: $x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and the forcing function $u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} 1 \\ t \end{bmatrix}$, $t \geq 0$.

The exact solution of the above MIMO state-space model is:

$$x(t)=\begin{bmatrix} exact_1 \\ exact_2 \end{bmatrix}=\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}=\begin{bmatrix} -6-2t+6e^{\frac{1}{2}t} \\ -5-t+e^{-t}+4e^{\frac{1}{2}t} \end{bmatrix}$$

Table (2) presents the comparison between the exact and numerical solution by applying (MM-SSES) algorithm for k=10, h=0.1 and $t_i = ih$, $i = 0,1,...,k$ depending on least square error (L.S.E.).

Table (2) Numerical results using Scraton method

t	Exact$_1$	Scraton $x_1(t)$	Exact$_2$	Scraton $x_2(t)$	Output $y_1(t)$	Scraton $y_1(t)$	Output $y_2(t)$	Scraton $y_2(t)$
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	0.1076	0.1076	0.0099	0.0099	0.1275	0.1275	0.1099	0.1099
0.2	0.2310	0.2310	0.0394	0.0394	0.3099	0.3099	0.2394	0.2394
0.3	0.3710	0.3710	0.0882	0.0882	0.5473	0.5473	0.3882	0.3882
0.4	0.5284	0.5284	0.1559	0.1559	0.8403	0.8403	0.5559	0.5559
0.5	0.7042	0.7042	0.2426	0.2426	1.1894	1.1894	0.7426	0.7426
0.6	0.8992	0.8992	0.3482	0.3482	1.5956	1.5956	0.9482	0.9482
0.7	1.1144	1.1144	0.4729	0.4729	2.0601	2.0601	1.1729	1.1729
0.8	1.3509	1.3509	0.6166	0.6166	2.5842	2.5842	1.4166	1.4166
0.9	1.6099	1.6099	0.7798	0.7798	3.1695	3.1695	1.6798	1.6798
1	1.8923	1.8923	0.9628	0.9628	3.8179	3.8179	1.9628	1.9628
L.S.E.		1.110e-14	L.S.E.	2.698e-14	L.S.E.	5.65e-14	L.S.E.	2.698e-14

Example (3) :

Consider the following SISO control system equation for the Pitch controller [9]:

$$\dddot{x}+6\ddot{x}+11\dot{x}+6x=\dddot{u}+8\ddot{u}+17\dot{u}+8u$$

with initial conditions: $y(0)=0, \dot{y}(0)=1, \ddot{y}(0)=0$ and the forcing function $u(t)=2t^3, t\geq 0$.

The state-space equation was derived using eq.(14) and eq.(15) as follows :

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix}=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}+\begin{bmatrix} 2 \\ -6 \\ 16 \end{bmatrix}[u(t)]$$

$$y(t)=\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}+u(t)$$

where the initial state of the state-space model is: $x(0)=\begin{bmatrix}x_1(0)\\x_2(0)\\x_3(0)\end{bmatrix}=\begin{bmatrix}0\\1\\0\end{bmatrix}$ and the forcing function $u(t)=2t^3,\ t\geq 0$.

The exact solution of the above SISO state-space model is:

$$x(t)=\begin{bmatrix}exact_1\\exact_2\\exact_3\end{bmatrix}=\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{bmatrix}=\begin{bmatrix}-\frac{5}{2}e^{-2t}+\frac{89}{54}e^{-3t}-\frac{19}{2}e^{-t}+\frac{2}{3}t^3+\frac{7}{3}t^2-\frac{77}{9}t+\frac{559}{54}\\5e^{-2t}-\frac{89}{18}e^{-3t}+\frac{19}{2}e^{-t}-4t^3+2t^2+\frac{14}{3}t-\frac{77}{9}\\-10e^{-2t}+\frac{89}{6}e^{-3t}-\frac{19}{2}e^{-t}+12t^3-12t^2+4t+\frac{14}{3}\end{bmatrix}$$

|Table (3) presents the comparison between the exact solution of state space model and numerical solution by applying (SS-SSES) algorithm for k=10, h=0.01 and $t_i=ih$, $i=0,1,...,k$ depending on (L.S.E.).

Table (3) Numerical results using Scraton method

t	Exact$_1$	Scraton $x_1(t)$	Exact$_2$	Scraton $x_2(t)$	Exact$_3$	Scraton $x_3(t)$	Output y(t)	Scraton y(t)
0	0.0000	0.0000	1.0000	1.0000	0.0000	0.0000	0.0000	0.0000
0.01	0.0100	0.0100	0.9995	0.9995	-0.1070	-0.1070	0.0100	0.0100
0.02	0.0200	0.0200	0.9979	0.9979	-0.2083	-0.2083	0.0200	0.0200
0.03	0.0300	0.0300	0.9953	0.9953	-0.3040	-0.3040	0.0300	0.0300
0.04	0.0399	0.0399	0.9918	0.9918	-0.3944	-0.3944	0.0400	0.0400
0.05	0.0498	0.0498	0.9874	0.9874	-0.4797	-0.4797	0.0500	0.0500
0.06	0.0596	0.0596	0.9822	0.9822	-0.5601	-0.5601	0.0601	0.0601
0.07	0.0695	0.0695	0.9762	0.9762	-0.6357	-0.6357	0.0701	0.0701
0.08	0.0792	0.0792	0.9694	0.9694	-0.7067	-0.7067	0.0802	0.0802
0.09	0.0889	0.0889	0.9619	0.9619	-0.7734	-0.7734	0.0903	0.0903
0.1	0.0985	0.0985	0.9538	0.9538	-0.8358	-0.8358	0.1005	0.1005
L.S.E.		1.259e-18	L.S.E.	2.491e-18	L.S.E.	2.194e-17	L.S.E.	0.159e-18

6. Conclusion:

Scraton method has been presented to find the numerical solution for different types of nth-order SSE's of linear continuous-time control system. The results show a marked improvement in the L.S.E. and the L.T.E. From solving some numerical examples the following points are included:

1- (SS-SSES) and (MM-SSES) algorithms gives a better accuracy and consistent to the solution of three types of nth-order state-space equations.

2- The accuracy of approximation in Scraton method depends on the size of h, if h is decreased then the number of knots increases and the (L.S.E. & L.T.E.) approaches zero.

7. References:

1. Ogata, Katsuhiko, “Modern Control Engineering”, Third Edition, Printed in the New Jersey, 1997.
2. Fairman , Frederick W. , "Linear Control Theory and The State Space Approach", John Wiley & Sons , Inc., USA 1998.
3. Vladimir Strejc , State Space Theory of Discrete Linear Control , Printed in Czechoslovak Academy of Sciences, 1989.
4. Amir Karniel and Gideon F. Inbar, “Linear System Description”, www.visl.technion.ac.il/karniel/pub/karnielInbar/ModernTechNs/ch21.pdf., 2010.
5. Lewis ,F.L., "State Variable Systems", www.cis.hut.fi/EE4314-controlsystems/lecture1.pdf. , 2005.
6. Masanao Aoki , "State Space Modeling of Time Series", Printed in Germany, Springer-Verlag , 1987.
7. Phillips ,Charles L. and Nagle ,H. Troy ,"Digital Control System Analysis and Design", Englewood Cliffs, N. J. ,Prentice Hall, Inc., 1992.
8. Wiberg, Donald M. ,"State Space and Linear Systems", McGraw-Hill ,Inc. in USA., 1981.
9. John, J.,"State Space Design for The Pitch Controller", www.krellinst.org/UCES/archive/classes/control/examples/pitch/invSS.htm1-144k, 2007.
10. James, Dk.,"State Space Control of The Cruise Control Model", www.krellinst.org/UCES/archive/classes/control/examples/cruise/ccSS.html-77k, 2011.
11. Lambert, J.D., “Computational Methods in Ordinary Differential Equations”, John Wiley & Sons Ltd., New York, 1999.
12. Shampin L.and Watts H.,” The Stability of The Scraton Methods”, J. Number Anal,Vol.13, pp. 101-114, 2003.

طريقة سكراتون لحل معادلات فضاء الحالة من الرتبة n لأنظمة السيطرة الخطية المستمرة الزمن

رغد كاظم صالح
قسم العلوم التطبيقية
الجامعة التكنولوجية

أثير جواد كاظم
قسم العلوم التطبيقية
الجامعة التكنولوجية

إسراء هادي حسن
قسم العلوم التطبيقية
الجامعة التكنولوجية

الخلاصة:

يقدم البحث طريقة مطورة مع خوارزميات مبرمجة بلغة ماتلاب لإيجاد الحل العددي لمعادلات فضاء الحالة الخطية من الرتبة n لأنظمة السيطرة المستمرة الزمن باستخدام طريقة سكراتون. حيث تمت معالجة أنظمة فضاء الحالة عدديا للمنظومات الفردية المدخل والمخرج مثلما للمنظومات المتعددة المداخل والمخارج باستخدام طريقة سكراتون من الرتبة الرابعة. كما تم توظيف طريقة سكراتون لايجاد النتائج العددية لقيم الإخراج لمعادلة فضاء الحالة . كما تمت مقارنة النتائج العددية و الحقيقية من خلال بعض الأمثلة لتوثيق دقة النتائج للطريقة المقترحة.