On gKc-Spaces

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Abstract

This paper is devoted to introduce new concepts so- called gKc-space, minimal gKc-space and locally gKc-space. Several various theorems about these concepts are proved. Further properties are stated as well as the relationships between these concepts with another types of KC-spaces are investigated.

Keywords: g-closed set, g-compact set, Kc-space.

1. Introduction:

It is known that compact subset of Hausdorff space is closed, this motivates the author [1] to introduce the concept of KC-spaces, and these are the spaces in which every compact subset is closed. In 2011 the authors [2] introduce new concepts namely K(gc) and gK(gc)-spaces. The aim of this paper is to continue the study KC-spaces.

2. Preliminaries:

The basic definitions that needed in this work are recalled. In this work, space X means a topological space (X, τ) on which no separation axioms are assumed, unless explicitly stated. The interior and the closure of any subset A of X will be denoted by Int(A) and cl(A) respectively. The author in [3] introduced the following definitions**:**

A subset F of a space X is called generalized closed (briefly g-closed) if cl(F)$\subseteq$O , whenever F$\subseteq$O and O is open in X. A subset A of a space X is called generalized open (briefly g-open), if and only if $F \subseteq$ Int(A) whenever F is closed and $F \subseteq A$. Also a space X is called $T_{1/2}$ -space if every g- closed set in X is closed. The author in [4] introduced the following definitions: A function f: X$\rightarrow$Y is said to be g-closed if f (F) is g-closed subset of Y, whenever F is closed subset of X, and is said to be g* -closed if f (F) is closed subset of Y, whenever F is g- closed subset of X, also is said to be g** -closed if f (F) is g-closed subset of Y, whenever F is g-closed subset of X . Also f is said to be g** -continuous if $f^{-1}(F)$ is g- closed (g-open) whenever F is g-closed (g-open) subset of Y. Also the g**-continuous image of g-compact is g-compact [2] and every g-closed subset of g-compact space is g-compact [5]. The author in [2] introduced the following **:**Let f be a homeomorphism function from a space X into space Y, if M is g-compact set in X, then f(M) is also g- compact. And if f is a homeomorphism function from a space X into space Y, and M is g-closed set in X, then f (M) is also g- closed.

Finally a space X is said to be gT_1 if for every two distinct points x and y, there exist two g-open sets U and V such that $x \in U$ and $y \notin U$, also $x \notin V$ and $y \in V$. The author in [5] introduced the following definition: A subset K of a space X is said to be generalized compact, (briefly g-compact) if for every g- open cover of K has a finite subcover. And The author in [6] introduced the following definition: A space X is said to be K_2 space if $cl(A)$ is compact, whenever A is compact set in X. Finally The author in [2] introduced the following definitions: A space X is said to be K(gc)or(gK (gc)) - space if every compact or(g-compact) set in X is g-closed.

A space X is said to be gK_2 space if $cl_g(A)$ is compact, whenever A is compact set in X.

3. The Kc-space:

In this section we introduce a generalization of Kc-spaces namely gKc-space, also we study the properties and facts about this concept and the relationships between this concept and gK(gc) and K(gc)-spaces.

First we introduce the following definition:

Definition3.1: A space X is said to be gKc if every g-compact set in X is closed. So the concept gKc-space is a generalization of Kc-space and a strong form of gK(gc)-spaces.

Example3.2: Let (R, τ_i) be the indiscrete topology on the real line R, then it is gK(gc)-space but not gKc, which implies it is not Kc-space. And also its K(gc)-space. Since if A is any compact proper subset of R, so the only open set which contain A is R, so cl(A)$\subset$R. Also it is not gKc- space. Since whenever A be g-compact proper subset of R then it is not necessarily is closed since the only closed sets are R and ϕ.

Definition3.3: A space X is said to be g^*K_2 if cl(A) is g-compact, whenever A is g- compact set in X.

Definition3.4: A space X is said to be $g^{**}K_2$- space if $cl_g(A)$is g-compact, whenever A is g-compact set in X.

Theorem3.5: Every gKc-space is g^*K_2- space.

Proof: Let W be g-compact set in gKc- space X. So W is closed, that is, cl(W)=W. Hence cl(W) is g-compact. Therefore X is g^*K_2.

Lemma3.6: Every gK(gc)-space is $g^{**}K_2$- space**.**

Proof: Let W be g-compact set in gK(gc)- space X. So W is g-closed, that is, $cl_g(W)$=W. Hence $cl_g(W)$ is g-compact. Therefore X is $g^{**}K_2$ space.

Theorem3.7: The g^{**}-continuous function f from g- compact space X into gKc – space Y is g^*-closed function.

Proof: Let F be g-closed in X, where X is g- compact space, then by lemma 3.8 F is g-compact but f is g^{**}-continuous function, then f(F) is g-compact in Y by lemma 3.7. But Y is gKc- space, which implies that f (F) is closed in Y. Hence f is g^*-closed function.

Corollary3.8: The g^{**}-continuous function f from g- compact space X into gKc –space Y is g^{**}-closed function.

Theorem3.9: Every continuous function from g-compact space X into Kc-space Y is g^*-closed.

Proof: Let F be g- closed subset of X, so F is g-compact in X by lemma 3.8. Since every g-compact is compact set, so F is compact subset of X. f is continuous function, then

f (F) is compact in Y where Y is Kc-space, which implies that f (F) is closed subset of Y. Hence f is g^* -closed function.

The prove of the following corollary is easy, hence is omitted.

Corollary3.10: Every continuous function from g-compact space X into K(gc)-space Y is g^{**}-closed.

Remark3.11: The continuous image of gKc-space may be not gkc- space, as in the following example:

Consider $I_R:(R,\tau_u)\to(R,\tau_i)$, where I_R the identity functions on R, (R,τ_u) is the usual topology on R and (R,τ_i) is the indiscrete topology on the real line R. (R,τ_u) is gKc-space, but (R,τ_i) is not gKc.

Theorem3.12: Let f be g^{**}-continuous injective function from X into gKc-space Y, then X is gK(gc).

Proof: Let F be any g- compact subset in X, then f(F)is g-compact in Y by lemma3.7. Since Y is gKc-space which implies that f(F) is closed in Y, so it is g- closed . But f is g^{**}-continuous injective function, so $f^{-1}(f(F))$ is g- closed in X. But F=$f^{-1}(f(F))$, therefore F is g-closed in X. Hence X is gK(gc).

Corallary3.13: Let f be g^{**}-continuous injective function from $T_{1/2}$ -space X into gKc-space Y, then X is gKc.

Lemma3.14: If X is $T_{1/2}$ -space and compact, then it is g- compact space.

Proof: Let X be a compact $T_{1/2}$ -space, to

Show that X is g-compact. Let $\{W_\alpha\}_{\alpha\in\Omega}$ be g- open cover of X, which is $T_{1/2}$, so every g- open set is open, and then $\{W_\alpha\}_{\alpha\in\Omega}$ be an open cover of X. But X is compact so every open cover reduces to a finite subcover.

Theorem3.15: If X is $T_{1/2}$ - space, then X is gKc if it is K(gc)-space.

Proof: Let M be g-compact subset of X, so it is compact. Since X is K(gc)-space, then M is g-closed in X and since X is $T_{1/2}$, then M is closed subset of X. Hence X is gKc –space.

Theorem3.16: If X is $T_{1/2}$ gKc, then X is K(gc)-space.

Proof: Let M be compact subset of $T_{1/2}$-space X, so it is g-compact by lemma 3.16. Since X is gKc-space, then M is closed in X so it is g- closed subset of X. Hence X is K(gc) –space.

Theorem3.17: Let Y be clopen subspace of X, then if W is g-open in X, then $W \cap Y$ is g-open in Y.

Proof: If $W \cap Y = \phi$, then $W \cap Y$ is g-open, since only closed subset of ϕ is itself and its subset of X, then $\phi^o = \phi$, since $\phi \in \tau$.

If $W \cap Y \neq \phi$, let F be closed subset of Y such that $F \subset W \cap Y$, to prove $F \subset Int(W \cap Y)_{inY}$. Since $F \subset W \cap Y$ then $F \subset W$ and $F \subset Y$. Because F is closed subset of Y and Y is closed subset in X, then F is closed subset of X. But W is g-open subset of X, which implies that $F \subset int(W)_{inX}$, where $F \subset W$. Since Y is open set of X and $F \subset Y$ so $F \subset Y = int(Y)_{inX}$. Therefore

$F \subset int(W)_{inX} \cap Y$.

But $int(W)_{inY} = int(W)_{inX} \cap Y$.

So $F \subset int(W)_{inY}$.

Since $Int(Y)_{inY} = Int(Y)_{inX} \cap Y$.

So $F \subset Int(Y)_{inY}$.

Therefore $F \subset int(W)_{inY} \cap int(Y)_{inY}$.

Which implies that $F \subset int(W \cap Y)_{inY}$.

Hence $W \cap Y$ is g-open in Y.

Theorem3.18: If $W \subseteq Y \subseteq X$. Let W be g-compact subset of Y and Y be clopen, subset of X, then W is g-compact subset of X.

Proof: Let W be g-compact subset of Y, to show W is g- compact subset of X. Let $\{U_\alpha\}_{\alpha \in \Omega}$ be g-open cover of W in X. $U'_\alpha = U_\alpha \cap Y$ is g-open in Y for each $\alpha \in \Omega$ by theorem3.19, that is, $\{U'_\alpha\}$ is g-open cover of W in Y. But W is g-compact in Y, then there exist a finite subcover of W in Y such that $W \subseteq \bigcup_{i=1}^{n} U'_{\alpha_i}$, then

$W \subseteq \bigcup_{i=1}^{n} (U_{\alpha_i})_{inX} \bigcap Y$.

Hence $W \subseteq \bigcup_{i=1}^{n} (U_{\alpha_i})_{inX}$.

Then W is g-compact in X.

Theorem3.19: The property of space being gKc is a hereditary property on clopen subspace.

Proof: Let Y be clopen subspace of gKc-space X, and A be any g-compact subset of Y, then its g-compact in X by theorem3.20.But X is gKc-space, then A is closed in X. But $Y \cap A = A$ is closed in Y , then A is closed in Y. Hence Y is gKc-space.

Theorem3.20: The property of space being gKc is a topological property.

Proof: Let $f: X \to Y$ be a homeomorphism function from a gKc –space X into space Y. Suppose F is a g-compact set in Y, then $f^{-1}(F)$ is g-compact in X by lemma 3.22. But X is gKc- space, then $f^{-1}(F)$ is closed in X, so f $(f^{-1}(F))$ is closed in Y.

But F= f $(f^{-1}(F))$, then F is closed in Y. Hence Y is gKc-space.

The prove of the following corollary is direct, hence is omitted.

Corollary3.21: Let $f:(X,\tau) \to (Y,\tau')$ be a homeomorphism function. Then if Y is gKc-space, then so is X.

Corollary3.22: The property of space being gK(gc) is a topological property.

Proof: Let $f: X \to Y$ be a homeomorphism function from a gK(gc) –space X into space Y. Suppose F is a g-compact set in Y, then $f^{-1}(F)$ is g-compact in X by lemma3.22.But X is gK(gc)- space, then $f^{-1}(F)$ is g-closed in X by lemma 3.25, so f($f^{-1}(F)$) is g-closed in Y.

Since F= f($f^{-1}(F)$), then F is g-closed in Y. Hence Y is gK(gc)-space.

Theorem3.23: Every T_2-space is gKc- space.

Proof: Let X be T_2-space and W be g-compact subset of X, so it is compact which implies that it is closed in X. Hence X is gKc- space.

Corolorry3.24: Every T_2-space is gK(gc)- space.

Theorem3.25: Every gKc- space is T_1-space.

Proof: Let $x \in X$, since {x} is finite set, then it is g- compact in X. But X is gKc- space, then {x} is closed.

Corolorry3.26: Every gKc- space is gT_1-space.

4. On Minimal gKc- spaces:

In this section we introduce a new concept namely minimal gKc- space.

Definition4.1: Let (X,τ) be gKc-space, then (X,τ) is said to be minimal gKc (or simply mgKc- space) if (X,τ^*) is not gKc-space where $\tau^* \subset \tau$.

We will use mgKc-space to denote the minimal gKc-space.

Remark: 4.2: Every mgKc- space is gKc, but the converse may be not true in general.

Theorem4.3: The property of space being mgKc is a topological property.

Proof: Let (X,τ_X) be a mgKc- space, and $f:(X,\tau_x) \to (Y,\tau_Y)$ be homeomorphism, and (Y, τ_Y) is gKc- space. To prove that Y is mgKc-space. Assume (Y, τ_Y) is not mgKc –space, then there exist a topology $\tau_Y^* \subset \tau_Y$ such that(Y, τ_Y^*) is a gKc-space. Define $\tau_1 = \{f^{-1}(V) : V \in \tau_Y^*\}$. τ_1 is a topology on X, $\tau_1 \subset \tau_x$ and (X,τ_1) is a gKc –space but that contract to the fact (X, τ_X) is mgKc-space. Hence (Y, τ_Y) is mgKc –space.

Remark4.4: The continuous image of mgKc-space may be not mgKc, as shown

in the following example:
Let $I_X : (X, \tau_D) \to (X, \tau_i)$,
be the identity function. (X, τ_D) is the discrete topology and (X, τ_i) is the indiscrete topology. (X, τ_D) is mgKc-space but (X, τ_i) is not mgKc-space, since it is not gKc.

Propostion4.5: Let (X, τ_X) be g-compact, gKc-space and (Y, τ_Y) be subspace of X. Then Y is g- compact if and only if Y is
g- closed in X.

Proof: Let (X, τ_X) be g-compact gKc-space, and (Y, τ_Y) be subspace of X. Suppose Y is g-compact in X and because X is gKc- space, then Y is closed in X so it is g-closed.
Conversely, suppose Y is g- closed in X, and since Y is g- compact, then Y is g-compact by lemma 3.8.

5. On locally gKc-space:

In this section we introduce a generalization of gKc-space namely locally gKc-spaces. We study the relationships between this generalization and gKc-spaces.

Defintion5.1: A space X is said to be locally gKc –space if each point in X has gKc neighborhood. So every gKc-space is locally gKc, but the converse is not true in general. In the next theorem we give the sufficient condition to make the converse is true.

Theorem5.2: A space X is an gKc-space if and only if each point has closed neighborhood which is an gKc- space.

Proof: If X is gKc-space, then for each $x \in X$, X itself is a closed neighborhood that is gKc.
Conversely, let L be g-compact in X such that $x \in X$ and $x \notin L$.
Choose a closed neighborhood W_x of x in X such that W_x is gKc- subspace in X. Then $W_x \cap L$ is closed in L and since L is g-compact, so $W_x \cap L$ is g-compact but W_x is gKc-space which implies that $W_x \cap L$ is closed in W_x.
Because W_x is closed in X, then $W_x \cap L$ is closed in X.
$W_x - (W_x \cap L) = W_x - L$, is a neighborhood of x disjoint from L. Hence L is closed in X. Therefore X is a gKc-space.

Theorem5.3: If a space X has the property that each point in X has open gKc-neighborhood, then X is T_1 –space.

Proof: Suppose X is not T_1, that is, there exist two distinct points x and y in X such that for every open set U contain y also contain x. Since $y \in X$, so there exists open gKc-neighborhood U of y so (U, τ_U) is T_1-space by theorem 3.29. Thus {y} is closed in U. Then U-{y} is open in U, but U is open in X, then U-{y}is open in X, but $y \notin U - \{y\}$ and $x \in U - \{y\}$ C! .Hence X is T_1- space.

Theorem5.4: If a space X has the property that each point in X has closed neighborhood which is gKc, then every clopen subspace is locally gKc.

Proof: Let Y be clopen subspace of X, where X has the property which state above, then X is gKc –space by theorem 5.2, so Y is also gKc-space by theorem 3.21. But every gKc-space is locally gKc, therefore Y is locally gKc.

Corrolary5.5: If X is gKc-space, then every clopen subspace of X is locally gKc.

Theorem 5.6: The property of space being locally gKc is a topological property

Proof: $f:(X,\tau_X)\to(Y,\tau_Y)$

be a homeomorphism function, where X is locally gKc-space. To show that Y is also locally gKc-space. Let $y\in Y$, then there exists $x\in X$ such that $f(x)=y$, also there exists a gKc neighborhood N of x.

Since f is homeomorphism then f(N) is also gKc neighborhood of y. Hence Y is also locally gKc-space.

Propsition5.7:A regular locally gKc-space is gKc.

Proof: Suppose X is a regular and locally gKc-space, then every point has a closed neighborhood which is gKc. Then by theorem 5.2, X is gKc.

Propsition5.8:

If X is a topological group, then X is gKc-space if and only if X is locally gKc –space.

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حول فضاءات gKc

الخلاصة

في هذا البحث قدمنا مفاهيم جديدة وهي الفضاءات gKc ،الفضاءات gKc الاصغرية و الفضاءات gKc المحلية.عدد من المبرهنات حول هذه المفاهيم قد برهنت. بالاضافة الى ذلك درسنا العلاقات والخصائص بين هذه الفضاءات المختلفة فيما بينها من جهة ومن جهة اخرى مع الفضاءات KC .