On S-Subcontinuous multifunctions

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Abstract.

In this paper we introduce a new concept, namely the s-subcontinuity for multifunction as a generalization of subcontinuity. We characterize s-compact preserving multifunctions in terms of s-subcontinuity. Conditions implying upper s-semi continuity for multifunction are derived.

Several new characterizations for upper s- semi-continuity and lower s- semi continuity are obtained using filterbasis. Furthermore we obtain some results on multifunction with s-closed graphs.

1. Introduction.

R.V. Fuller in [5] introduced the concept of subcontinuous function and used it to obtain conditions implying continuity. In [9] R. E. Simthson extended this concept to multifunctions and used it to obtain a number of results on multifunctions, and also developed criteria under which a multifunction is upper semi continuous.

Let A be a subset of X, the closure of A and the interior of A are denoted by $Cl(A)$ and $Int(A)$ respectively . Levin in [6] introduced the notion of semi – open sets (briefly s-open), so that A is s-open if there exists an open set U such that $U \subset A \subset Cl(U)$, equivalently $A \subset Cl(Int(A))$. The family of all s-open sets is denoted by $SO(X)$. The complement of s-open set is said to be semi-closed (briefly s-closed), the family of all s-closed sets of X is denoted by $SC(X)$. The smallest s-closed set containing a subset A is called the semi-closure of A and denoted by $sCl(A)$ [3]. The semi-interior of A denoted by $sInt(A)$, is the largest s-open set contained in A. A subset $A \subset X$ is called α-set if $A \subset Int(Cl(Int(A)))$ [7]. The family of all α-sets in X is denoted by τ^{α}. It was shown that τ^{α} is a topology on X finer than τ [7]. A point $x \in X$ is called semi limit(s-limit) point of $A \subset X$ if every s-open set containing x contains a point of A different from x [2]. A subset N of a space X is called semi-neighborhood (s-nbd) of a point $x \in X$ iff there exists $U \in SO(X)$ such $x \in U \subset N$. A net $\{x_n\}$ in X is called s-convergent to $x \in X$, denoted by $\{x_n\} \xrightarrow{s} x$ iff $\{x_n\}$ is eventually in every semi-open set containing x [2]. If Ω is a filterbase on X then we define the semi-adherence of Ω in X to be the set $\cap\{sCl(B):B \in \Omega\}$ and denote it by

$ad_s\,\Omega$.

Throughout the present paper $F : X \to Y$ (respectively $f : X \to Y$) represents a multifunction (respectively a single valued function). For a multifunction $F : X \to Y$ the upper and lower inverse of a subset V of Y are denoted by $F^{+}(V)$ and $F^{-}(V)$ respectively, where $F^{+}(V) = \{x \in X : F(x) \subset V\}$ and $F^{-}(V) = \{x \in X : F(x) \cap V \neq \phi\}$. Further, if $A \subset X$ then $F(A) = \bigcup\{F(x) : x \in A\}$.We will denote the graph of F by G_F Where $G_F = \{(x, y) : x \in X, y \in F(x)\}$. A multifunction F is said to have closed (s-closed) graph if G_F is closed (s-closed) subset of the space $X \times Y$. Let P be a property of sets, then a multifunction $F : X \to Y$ is called point P if F(x) has property P for each $x \in X$. Properties we shall use in this paper are closed, s-closed, and s- rigid (Definition 3.1).

Throughout this paper X and Y represent nonempty topological spaces on which no separation axioms are assumed unless otherwise mentioned.

2. Preliminaries

The following definitions and results will be useful in the sequel.

2.1. Theorem. [2]

If $\{x_n\}$ is s-convergent net then $\{x_n\}$ is convergent and the converse is not necessarily true as the following example shows:

2.2. example. [2]

let X = [-1,1] with usual topology on X , then $\{\frac{1}{n}\}_{n \in N}$ converges to 0 but $\{\frac{1}{n}\}_{n \in N}$ not s-converges to 0 .

2.3. Definition. [2]

A space X is semi compact (briefly s-compact) iff every cover by s-open sets has a finite subcover, and $A \subset X$ is s-compact iff every cover of A by s-open sets in A has a finite subcover.

2.4. Theorem. [2]

A space X is s-compact if and only if every net in X has an s-convergent subset.

The following theorem is easy to prove,

2.5. Theorem

A space X is s-compact if and only if every s-closed subset of X is s-compact.

2.6. Definition.[7]

A space X is called extremely disconnected if the closure of each open set in X is open in X .

We introduce the following result which is a consequence.

2.7. Theorem.

Let X be extremely disconnected, $x \in X$ and $A \subset X$. Then $x \in sCl(A)$ iff there is a net of points of *A*, s-converging to *x*.

2.8. Theorem. [3]

$Int(Cl(A)) \subset sInt(sCl(A))$.

Using this theorem and the fact that $sInt(A) \subset A$, we have the following result.

2.9. Lemma.

$Int(Cl(A)) \subset sCl(A))$.

2.11. Definition. [1]

A multifunction $F : X \to Y$ is said to be:

(1) Upper semi continuous $(u.s.c)$ if for each $x \in X$ and each open set *V* of *Y* containing *F (x)* there exists an open set $U \subset X$ containing *x* such that $F(U) \subset V$.

(2) Lower semi continuous $(l.s.c.)$ if for each $x \in X$ and each open set *V* of *Y* such that $F(x) \cap V \neq \phi$ there exists an open set $U \subset X$ containing *x* such that $F(u) \cap V \neq \phi$ for every $u \in U$.

(3) Continuous if it is $(u.s.c)$ and $(l.s.c.)$.

2.12. Definition. [8]

A multifunction $F : X \to Y$ is said to be:

(1) Upper s-semi continuous $(\bar{s}.s.c.)$ if $F^{+}(V) \in SO(X)$ for each open set V of Y.

(2) Lower s-semi continuous $(\underline{s}\ s.c.)$ if $F^{-}(V) \in SO(X)$ for each open set V of Y.

(3) Semi-continuous $(s.c.)$ if it is $(\bar{s}.s.c.)$ and $(\underline{s}.\ s.c.)$.

The $(\bar{s}.s.c.)$ and $(\underline{s}.s.c.)$ multifunctions was studied in some details by Popa [8].

2.13. Theorem. [8]

The following are equivalent for multifunction $F : X \to Y$.

(1) F is $(\bar{s}.s.c.)$

(2) For each $x \in X$ and each open set $V \subset Y$ with $F(x) \subset V$ there exist $U \in SO(X)$ such that $x \in U$ and $F(U) \subset V$.

(3) $F^{-}(V) \in SC(X)$ for each closed set $V \subset Y$.

(4) $Int(Cl(F^{-}(B))) \subset F^{-}(Cl(B))$ for each $B \subset Y$.

3. Some Characterizations.

In this section we introduce the notion of s-rigidity analogous to Dickman's definition of θ-rigidity [4] and introduce new properties and characterizations for $(\bar{s}.s.c.)$ and $(\underline{s}\ s.c.)$ multifunctions.

3.1. Definition.

A set $A \subset X$ is s-rigid if $A \cap ad_s\ \Omega \neq \phi$ whenever Ω is a filterbase on X satisfying $B \cap U \neq \phi$ for each $B \in \Omega$ and $U \in SO(X)$ containing A .

The following Lemma is needed in the proofs of the next results.

3.2. Lemma.

Let $A \subset X$, $x \in X$, then, x$\in$ *sCl (A)* if and only if for each $U \in SO(X)$ containing *x*, $U \cap A \neq \phi$.

Proof:

Suppose that x is not in *sCl (A)* , the set *U = X- sCl (A)* is an s-open set containing x and $U \cap A = \phi$. Conversely, if there is an s- open set U containing x such that $U \cap A \neq \phi$, then X- U is an s- closed set containing A. hence X- U contain *sCl (A),* therefore x cannot in *sCl (A).*

Now we introduce the following characterizations for $\bar{s}.s.c.$multifunctions.

3.3. Theorem.

The following statements are equivalent for a multifunction $F : X \to Y$, where F is point s- rigid.

(1) F is $\bar{s}.s.c.$

(2) $ad_s F^-(\Omega) \subset F^-(ad_s \Omega)$ for each filterbase Ω on $F(X)$.

(3) $sCl(F^-(B)) \subset F^-(sCl(B))$. for each $B \subset Y$.

Proof:

(1)$\Rightarrow$(2) Let $x \in ad_s F^-(\Omega)$. Since F is $(\bar{s}.s.c.)$, for each open set W in Y such that $F(x) \subset W$, there is $U \in SO(X)$ containing x such that $F(U) \subset W$. Since $x \in ad_s F^-(\Omega)$, so $x \in sCl(F^-(A))$ for each $A \in \Omega$. Therefore, by Lemma 3.2, $F^-(A) \cap U \neq \phi$, thus $A \cap W \neq \phi$. Since *F(x)* is s-rigid, it follows that $F(x) \cap ad_s\ \Omega \neq \phi$. Hence $x \cap F^-(ad_s\ \Omega)$.

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1): Obvious.

Popa in [8] proved several characterizations for multifunction. Our next theorem improves on these results by using filterbases, thus (3), (4), (5) are new characterizations.

3.4. Theorem.

The following are equivalent for a multifunction $F : X \to Y$.

(1) F is $(\underline{s}\ s.c.)$.

(2) For each open set $V \subset Y$ and for each x X with $F(x) \cap V$, there is $U \in SO(X)$ containing x and $F(u) \cap V$ for each $u \in U$.

(3) $F(ad_s\ \Omega)\ ad\ F(\Omega)$ for each filterbase Ω on X.

(4) $F(sCl(A)) \subset Cl((F(A))$ for each $A \subset X$.

(5) Each family Ω of subsets of Y satisfies

$\bigcap_{B\in\Omega} sCl(F^+(B)) \subset F^+(\bigcap_{B\in\Omega} Cl(B))$

(6) $sCl(F^+(B)) \subset F^+(Cl(B))$ for each $B \subset Y$.

(7) $F^{+}(B)$ is s-closed in X, for each $B \subset Y$.
(8) Int $(F^{+}(B)) \subset F^{+}(Cl(B))$ for each $B \subset Y$.

Proof:

(1) $\Rightarrow$ (2) This is proved in [8].

(2) $\Rightarrow$ (3) Let $x \in ad_s\,\Omega$ and $y \in F(x)$. Let W be open in Y such that $y \in W$. Hence by (2) there is $U \in SO(X)$ containing x such that $U \subset F^{-}(W)$. Since $x \in ad_s\,\Omega$, so $x \in sCl(A)$ for each $A \in \Omega$. Hence, by Lemma 3.2, for each $U \in SO(X)$ containing x, $U \cap A \neq \phi$. Therefore $F^{-}(W) \cap A \neq \phi$. Thus $W \cap F(A) \neq \phi$ for each $A \in \Omega$, so $y \in Cl(F(A))$ for each $F(A) \in F(\Omega)$. Therefore $y \in ad\, F(\Omega)$.

(3)$\Rightarrow$(4): Obvious.

(4)$\Rightarrow$(5): Suppose Ω is a family of subset of Y which fails to satisfy the inequality in (5). Thus $\bigcap_{B\in\Omega} sCl(F^{+}(B)) \not\subset F^{+}(\bigcap_{B\in\Omega} Cl(B))$, so there is $x \in \bigcap_{B\in\Omega} sCl(F^{+}(B))$ and $x \notin F^{+}(\bigcap_{B\in\Omega} Cl(B))$. Hence $F(x) \subset F(sCl(F^{+}(B))$ for each $B \in \Omega$, and $F(x) \not\subset Cl(B)$ for some $B \in \Omega$. Hence $F(x) \not\subset Cl(F(F^{+}(B)))$.
Therefore $F(sCl(F^{+}(B)) \not\subset Cl(F(F^{+}(B)))$, thus (4) fails.

(5)$\Rightarrow$(6): This is obvious .

(6)$\Rightarrow$(7), (7)$\Rightarrow$(8) and (8)$\Rightarrow$(1): These are proved in [8].

4. S-Subcontinuity and Upper S- Semi Continuity.

Fuller [5] introduced and studied the notion of subcontinuous function, Smithson [9] extended this definition to multifunction, so that $F: X \rightarrow Y$ is subcontinuous iff whenever $\{x_n\}$ is convergent net in X, and $\{y_n\}$ is a net in $F(X)$ with $y_n \in F(x_n)$ for each n, then $\{y_n\}$ has a convergent subnet.

We introduce the notion of s-subcontinuity as a generalization of Simthson's definition.

4.1 Definition.

A multifunction $F: X \rightarrow Y$ is s-subcontinuous if and only if whenever $\{x_n\}$ is s-convergent net in X and $\{y_n\}$ is a net in $F(X)$ with $y_n \in F(x_n)$ for each n, then $\{ y_n \}$ has an s-convergent subset.

The following result is an immediate consequence of definition 4.1 and Theorems 2.4, 2.5.

4.2 Theorem.

Let $F: X \rightarrow Y$ be an s-subcontinuous multifunction if $A \subset X$ is s-compact and $F(A)$ is s-closed then $F(A)$ is s-compact.

Now we characterize s-compact preserving multifunctions in terms of s-subcontinuity.

4.3 Theorem.

Let $F: X \to Y$ be a multifunction. Then *F* is s-compact preserving iff $F_{|A} \to F(A)$ is s-subcontinuous for each s-compact $A \subset X$, where $F_{|A}$ denotes the restriction of F on A.

Proof:

Necessity: suppose *F* is an s-compact preserving. Let *A* be s-compact, then *F (A)* is s-compact. Then by Theorem 2.4, every net in *F (A)* has an s-convergent subnet. Hence $F_{|A} \to F(A)$ is s-subcontinuous.

Sufficiency: let $A \subset X$ be s-compact and $F_{|A} \to F(A)$ be s-subcontinuous. Then every net in *F (A)* have a convergent subnet. Hence by Theorem 2.4, *F (A)* is s-compact .

Now we use the notion of closed graph to give some properties of $\bar{s}.s.c.$ multifunctions, first we introduce the following example to show that a multifunction with an s-closed graph need not be s-continuous.

4.4. Example.

Let $X=\{a, b, c, d\}$, $Y=\{1,2,3\}$ with topologies $\tau=\{x, \phi, \{a\}\}$, $J=\{y, \phi, \{2\}\}$ respectively, then.

$SO(X) = \{X, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, d\}, \{a, d, b\}, \{a, d, c\}\}$

$SO(Y) = \{y, \phi, \{1,2\}, \{2\}, \{2,3\}\}$, define $F: X \to Y$ by $F(a) = \{1\}$, $F(b) = \{3\}$, $F(c) = F(d) = \{2\}$ then F has s-closed graph but *F* is not s-continuous.

In the following result we give sufficient conditions for a multifunction with s- closed graph to be $\bar{s}.s.c.$.

4.5. Theorem.

Let *X* be extremely disconnected space and $F: X \to Y$ be s-subcontinuous multifunction which has an s-closed graph. Then *F* is $\bar{s}.s.c.$.

Proof:

Let $B \subset Y$ be closed and $x_0 \in sCl(F^{-}(B))$, then by Theorem 2.7, there is a net $\{x_n\}$ in $F^{-}(B)$ which s-converges to x_0 . Let $\{y_n\}$ be a net in *B* such that $y_n \in F(x_n)$ for each n. Since *F* is s- subcontinuous, there is an s-convergent subnet $y_{n_m} \xrightarrow{s} y_0 \in B$. If $y_0 \notin F(x_0)$ then $(x_0, y_0) \notin G_F$, but G_F is s-closed, so there are s-open sets $U \subset X$ and $V \subset Y$ such that $(x_0, y_0) \in U \times V$ and $(U \times V) \cap G_F = \phi$, since $x_n \xrightarrow{s} x_0$, and $y_{n_m} \xrightarrow{s} y_0$. Hence by Theorem 2.1, $x_n \to x_0$, and $y_{n_m} \to y_0$. Thus there is n_m such that $x_{n_m} \in U$ and $y_{m_n} \in V$ which is a contradiction. Thus $y_0 \in F(x_0)$ and $x_0 \in F^{-}(A)$. Hence *F* is $\bar{s}.s.c.$

The following result is obtained from the proof of the above theorem.

4.6. Corollary.

Let *X* be extremely disconnected space and $F: X \to Y$ be an s-subcontinuous multifunction with s-closed graph . Let $x_n \xrightarrow{s} x_0$ and

$\{y_n\}$ be a net such that $y_n \in F(x_n)$ for each n . If $y_n \xrightarrow{s} y_0$ then $y_0 \in F(x_0)$.

If we assume that Y is regular in Theorem 4.5, then we have the following result.

4.7. Theorem.

If $F: X \to Y$ is a point closed $\bar{s}.s.c.$ and Y is a regular space. Then F has an s-closed graph.

proof:

Suppose $(x, y) \notin G_F$ then $y \notin F(x)$. But F is point closed, so $F(x)$ is closed. Since Y is regular, there are U, V open in Y such that $y \in U$, $F(x) \subset V$ and $U \cap V = \phi$. Since F is $(\bar{s}.s.c.)$, there is $W \in SO(X)$ such that $x \in W$ and $F(W) \subset V$. Thus $(x, y) \in W \times U$ and $(W \times U) \cap G_F = \phi$. Hence $(x, y) \notin sCl(G_F)$. Therefore G_F is s-closed.

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