

First and Second Chebyshev Wavelets Spectral Method for Solving Variational Problems

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Abstract:

In this paper, Chebyshev Operational matrices of Integration (first and second) for the approximate solutions of Variational problems are introduced.

The main idea behind this approach is that it reduced such problems to ones of solving systems of algebraic equations using spectral method. Only a small number of Chebyshev wavelets are needed to obtain a satisfactory results. The applications of the proposed method are demonstrated through illustrative examples.

Keywords: Chebyshev wavelets, Operational matrix of integration, Spectral method.

1. Introduction:

Many different basic functions have been used to solve variational problems, such as orthogonal functions and wavelets [3,7,8].

The applications of wavelets as basic functions in the numerical solutions of many problems are widely used in the last years.

Typical examples are the Legendre wavelets for solving linear Fredholm and Volterra integral equation of the second kind [6], Haar wavelets for solving initial and boundary value problems of Bratu-type [2], Chebyshev wavelets for solving Abel integral equation [7].

In this paper, Chebyshev wavelets basis of first and second kinds, on the interval [0,1] have been considered for solving the following Variational Problems.

$$J(x) = \int_0^1 F(t, x(t), \dot{x}(t)) dt \quad \dots(1)$$

Thus are some applications of Chebyshev wavelets method in the literature [1]. Some techniques have been used for solving Variational problems.

2. Chebyshev Wavelets and Their Properties

2.1 The Definition of First Chebyshev Wavelets [1]

First Chebyshev wavelet $\Psi_{n,m}^1(t) = \Psi_{(t,n,m,k)}^1$ have four arguments; $k = 1, 2, \dots$, $n = 1, 2, \dots, 2^k$, m , is order for first kind Chebyshev polynomials and t is the normalized time, they are defined on the interval [0,1) by

$$\Psi_{n,m}^1(t) = \begin{cases} \frac{c_m 2^{\frac{n-1}{2}}}{\sqrt{\pi}} T_m(2^{k+1}t - 2n + 1) & \frac{n-1}{2^k} \leq t < \frac{n}{2^k} \\ 0 & \text{o.w} \end{cases} \quad \dots(2)$$

where $c_m = \begin{cases} \sqrt{2} & m = 0 \\ 2 & m = 1, 2, \dots \end{cases}$

We known T_m is orthogonal with respect to the weight function $w(t) \frac{1}{\sqrt{1-t^2}}$ the set of Chebyshev wavelets are orthogonal with respect to weight function $w_n(t) = w(2^{k+1}t - 2n + 1)$

2.2 The Definition of Second Chebyshev Wavelets [4]

The second Chebyshev wavelets $\Psi^2(t)$, $\Psi_{nm}^2(t) = \Psi^2(t, n, m, k)$ involve four arguments; $k = 1, 2, \dots$, $n = 1, 2, \dots, 2^{k-1}$, m is the degree of the second Chebyshev polynomials and t is the normalized time.

They are defined on the interval $[0, 1)$ as

$$\Psi_{nm}^2(t) = \begin{cases} 2^{\frac{k}{2}} \tilde{U}_m(2^k t - 2n + 1) & \\ 0 & \end{cases} \quad \dots(3)$$

where $\tilde{U}_m(t) = \sqrt{\frac{2}{\pi}} U_m(t)$... (4)

and $m = 0, 1, 2, \dots, M-1$ in equation (3) we know the second Chebyshev polynomials is orthogonal with respect weight function $w(t) = \sqrt{1-t^2}$ on the interval $[-1, 1]$, we should note that in dealing with the second Chebyshev wavelets the weight function $\bar{w}(t) = w(2^k t - 2n + 1)$

3. Function Approximation

A function approximation $f(t) \in L_2[0, 1)$ may be expanded as

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} C_{nm} \Psi_{nm}(t) \quad \dots(5)$$

where $C_{nm} = (f(t), \Psi_{nm}(t))$... (6)

In equation (4), $(\cdot, \cdot)$ denoted the inner product with weight function $w_n(t)$ on the Hilbert Space $[0, 1)$

If the infinite series in above equation is truncated, then equation (5) can be written as:

$$f(t) = \sum_{n=1}^{2^k} \sum_{m=0}^{M-1} C_{nm} \Psi_{nm}(t) = C^T \Psi_{nm}(t). \quad \dots(7)$$

where C and $\Psi_{nm}(t)$ matrices are given by:

• First Chebyshev Wavelets:

$$C = [C_{10}, C_{11}, \dots, C_{1(M-1)}, C_{20}, \dots, C_{2(M-1)}, \dots, C_{2^k}, \dots, C_{2^k M-1}]^T. \quad \dots(8)$$

$$\Psi_{nm} = \Psi^1(t) = [\Psi_{10}^1(t), \Psi_{11}^1(t), \dots, \Psi_{1(M-1)}^1(t), \Psi_{20}^1(t), \dots, \Psi_{2(M-1)}^1(t), \dots, \Psi_{2^k 0}^1(t), \dots, \Psi_{2^k M-1}^1(t)]^T. \quad (9)$$

where C and Ψ are $2^k M \times 1$ matrices.

• Second Chebyshev Wavelets

$$C = [C_{10}, C_{11}, \dots, C_{1(M-1)}, C_{20}, \dots, C_{2(M-1)}, \dots, C_{2^{k-1} 0}, \dots, C_{2^{k-1} M-1}]^T. \quad \dots(10)$$

$$\Psi_{nm} = \Psi^2(t) = [\Psi_{10}^2(t), \Psi_{11}^2(t), \dots, \Psi_{1(M-1)}^2(t), \Psi_{20}^2(t), \dots, \Psi_{2(M-1)}^2(t), \dots, \Psi_{2^{k-1} 0}^2(t), \dots, \Psi_{2^{k-1} M-1}^2(t)]^T \quad \dots(11)$$

where C and Ψ are $2^{k-1} M \times 1$ matrices.

4. Operational Matrix of Integration

In general, the operational matrix of integration may be defined,

$$\int_0^t \Psi(t) = P \Psi(t) \quad \dots(12)$$

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where $\tau(t)$ is the vector of basis function, P is the operational matrix of integration

Higher order integrals may be found by using equation (12) recursively.

Thus the n th integral may be written as $\int_0^t \dots \int_0^t \Psi(t) dt = P^n \Psi(t)$ (13)

4.1 Operational Matrix of Integration for First Chebyshev Wavelets

The integration of the vector $\Psi^1(t)$, defined in (9), can be achieved as

$$\int_0^t \Psi^1(t) dt = P_{\Psi^1} \Psi^1(t)$$

where P_{Ψ^1} is the $(2^k M) \times (2^k M)$ operational matrix of integration [2]. This matrix is determined as follows.

$$P_{\Psi^1} = \begin{bmatrix} C & S & S & \dots & S \\ 0 & C & S & \dots & S \\ 0 & 0 & C & \ddots & S \\ \vdots & \vdots & \vdots & \ddots & S \\ 0 & 0 & 0 & \dots & C \end{bmatrix},$$

where S and C are $M \times M$ matrices given

$$S = \frac{\sqrt{2}}{2^k} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ -\frac{1}{3} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ -\frac{1}{15} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{M(M-2)} & 0 & 0 & \dots & 0 \end{bmatrix}$$

$$C = \frac{1}{2^k} \begin{bmatrix} \frac{1}{2} & \frac{1}{2\sqrt{2}} & 0 & 0 & \dots & 0 & 0 & 0 \\ -\frac{1}{4\sqrt{2}} & 0 & \frac{1}{8} & 0 & \dots & 0 & 0 & 0 \\ -\frac{1}{3\sqrt{2}} & -\frac{1}{4} & 0 & \frac{1}{12} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -\frac{1}{2\sqrt{2}(M-1)(M-3)} & 0 & 0 & 0 & \dots & -\frac{1}{4(M-3)} & 0 & -\frac{1}{4(M-1)} \\ -\frac{1}{2\sqrt{2}M(M-2)} & 0 & 0 & 0 & \dots & 0 & -\frac{1}{4(M-2)} & 0 \end{bmatrix}$$

4.2 Operational Matrix of Integration for Second Chebyshev Wavelets

The integration of the vector $\Psi^2(t)$, defined in (11), can be achieved as

$$\int_0^t \Psi^2(t) dt = P_{\Psi^1} \Psi^2(t)$$

where P_{Ψ^2} is the $(2^{k-1}M) \times (2^{k-1}M)$ operational matrix of integration [3]. This matrix is determined as follows.

$$P_{\Psi^2} = \frac{1}{2^k} \begin{bmatrix} L & S & S & \dots & S \\ 0 & L & S & \dots & S \\ 0 & 0 & L & \dots & S \\ \vdots & \vdots & \vdots & \ddots & S \\ 0 & 0 & 0 & \dots & L \end{bmatrix}$$

where S and L are $M \times M$ matrices as follows:

$$S = \begin{bmatrix} 2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \quad L = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 \\ -\frac{3}{4} & 0 & \frac{1}{4} & \dots & 0 \\ \frac{1}{3} & -\frac{1}{6} & 0 & \dots & 0 \\ -\frac{1}{4} & 0 & -\frac{1}{8} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1^{M-2} \frac{1}{M-1} & 0 & 0 & \dots & \frac{1}{2(M-1)} \\ -1^{M-1} \frac{1}{M} & 0 & 0 & \dots & 0 \end{bmatrix}$$

5. Powers in terms Chebyshev wavelets

5.1 Powers in terms of First Chebyshev wavelets

For $k = 1, n = 1, 2, \dots, M = 1, 2, \dots, 6, \dots$ and t are the normalized time, we will drive the powers in terms of first Chebyshev wavelets, which help us to solve our problems. Let's take $M = 6$ and $m = 0, 1, 2, \dots, 6$. First, find fourteen basis functions are given by

$$\left. \begin{aligned} \Psi_{1,0}^1 &= \frac{2}{\sqrt{\pi}} \\ \Psi_{1,1}^1 &= \frac{2\sqrt{2}}{\sqrt{\pi}} (4t - 1) \\ \Psi_{1,2}^1 &= \frac{2\sqrt{2}}{\sqrt{\pi}} (32t^2 - 16t + 1) \\ \Psi_{1,3}^1 &= \frac{2\sqrt{2}}{\sqrt{\pi}} (256t^3 - 192t^2 + 36t - 1) \\ \Psi_{1,4}^1 &= \frac{2\sqrt{2}}{\sqrt{\pi}} (2048t^4 - 2048t^3 + 640t^2 - 64t + 1) \\ \Psi_{1,5}^1 &= \frac{2\sqrt{2}}{\sqrt{\pi}} (16384t^5 - 20480t^4 + 8960t^3 - 1600t^2 + 100t - 1) \\ \Psi_{1,6}^1 &= \frac{2\sqrt{2}}{\sqrt{\pi}} (131072t^6 - 196608t^5 + 110592t^4 - 28672t^3 + 3360t^2 - 144t + 1) \end{aligned} \right\} \text{where } 0 \leq t < \frac{1}{2}$$

$$\left. \begin{aligned} \Psi_{2,0}^1 &= \frac{2}{\sqrt{2}} \\ \Psi_{2,1}^1 &= \frac{2\sqrt{2}}{\sqrt{2}} (4t - 3) \\ \Psi_{2,2}^1 &= \frac{2\sqrt{2}}{\sqrt{2}} (32t^2 - 48t + 17) \\ \Psi_{2,3}^1 &= \frac{2\sqrt{2}}{\sqrt{2}} (256t^3 - 576t^2 + 420t - 99) \\ \Psi_{2,4}^1 &= \frac{2\sqrt{2}}{\sqrt{2}} (2048t^4 - 6144t^3 + 6784t^2 - 3264t + 577) \\ \Psi_{2,5}^1 &= \frac{2\sqrt{2}}{\sqrt{2}} (16384t^5 - 61440t^4 + 90880t^3 - 66240t^2 + 23780t - 3363) \\ \Psi_{2,6}^1 &= \frac{2\sqrt{2}}{\sqrt{2}} (131072t^6 - 589824t^5 + 1093632t^4 - 1069056t^3 + 580896t^2 - 166320t + 19601) \end{aligned} \right\} \text{where } \frac{1}{2} \leq t < 1$$

In matrix form, the powers of t can be rewritten as follows

$$A = T_{\Psi^1} B$$

and $T_{\psi^1} = \frac{\sqrt{\pi}}{2\sqrt{2}}$

$\frac{\sqrt{2}}{4}$	0	0	0	0	0	0
$\frac{\sqrt{2}}{32}$	$\frac{1}{4}$	0	0	0	0	0
$\frac{10\sqrt{2}}{256}$	$\frac{4}{32}$	$\frac{1}{32}$	0	0	0	0
$\frac{35\sqrt{2}}{2048}$	$\frac{15}{256}$	$\frac{6}{256}$	$\frac{1}{256}$	0	0	0
$\frac{126\sqrt{2}}{16384}$	$\frac{56}{2048}$	$\frac{28}{2048}$	$\frac{8}{2048}$	$\frac{1}{2048}$	0	0
$\frac{462\sqrt{2}}{131072}$	$\frac{210}{16384}$	$\frac{120}{16384}$	$\frac{45}{16384}$	$\frac{10}{16384}$	$\frac{1}{16384}$	0
	$\frac{792}{131072}$	$\frac{495}{131072}$	$\frac{220}{131072}$	$\frac{66}{131072}$	$\frac{12}{131072}$	$\frac{1}{131072}$

if $0 \leq t < \frac{1}{2}$

$T_{\psi^1} = \frac{\sqrt{\pi}}{2\sqrt{2}}$

$\frac{\sqrt{2}}{4}$	0	0	0	0	0	0
$\frac{3\sqrt{2}}{32}$	$\frac{1}{4}$	0	0	0	0	0
$\frac{19\sqrt{2}}{256}$	$\frac{12}{32}$	$\frac{1}{32}$	0	0	0	0
$\frac{126\sqrt{2}}{2048}$	$\frac{111}{256}$	$\frac{18}{256}$	$\frac{1}{256}$	0	0	0
$\frac{867\sqrt{2}}{16384}$	$\frac{936}{2048}$	$\frac{220}{2048}$	$\frac{24}{2048}$	$\frac{1}{2048}$	0	0
$\frac{6138\sqrt{2}}{131072}$	$\frac{7570}{16384}$	$\frac{2280}{16384}$	$\frac{365}{16384}$	$\frac{30}{16384}$	$\frac{1}{16384}$	0
	$\frac{59976}{131072}$	$\frac{21615}{131072}$	$\frac{4500}{131072}$	$\frac{546}{131072}$	$\frac{36}{131072}$	$\frac{1}{131072}$

if $\frac{1}{2} \leq t < 1$

$$\text{Where } A = \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \\ t^4 \\ t^5 \\ t^6 \end{pmatrix}, \quad B = \begin{pmatrix} \Psi_{1,0}^1 \\ \Psi_{1,1}^1 \\ \Psi_{1,2}^1 \\ \Psi_{1,3}^1 \\ \Psi_{1,4}^1 \\ \Psi_{1,5}^1 \\ \Psi_{1,6}^1 \end{pmatrix}$$

5.2 Powers in terms of second Chebyshev wavelets:

Similarly we will drive above the powers in terms of second Chebyshev wavelets, for

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$k = 2$ again, we find 14×14 matrix of powers. The fourteen basis functions are given by

$$\left. \begin{aligned} \psi_{1,0}^2 &= \frac{2\sqrt{2}}{\sqrt{\pi}} \\ \psi_{1,1}^2 &= \frac{2\sqrt{2}}{\sqrt{\pi}} (8t - 2) \\ \psi_{1,2}^2 &= \frac{2\sqrt{2}}{\sqrt{\pi}} (64t^2 - 32t + 3) \\ \psi_{1,3}^2 &= \frac{2\sqrt{2}}{\sqrt{\pi}} (512t^3 - 384t^2 + 80t - 4) \\ \psi_{1,4}^2 &= \frac{2\sqrt{2}}{\sqrt{\pi}} (4096t^4 - 4096t^3 + 1344t^2 - 160t + 5) \\ \psi_{1,5}^2 &= \frac{2\sqrt{2}}{\sqrt{\pi}} (32768t^5 - 40960t^4 + 18432t^3 - 3584t^2 + 280t - 6) \\ \psi_{1,6}^2 &= \frac{2\sqrt{2}}{\sqrt{\pi}} (262144t^6 - 393216t^5 + 225280t^4 - 61440t^3 + 8064t^2 - 448t + 7) \end{aligned} \right\} \text{where } 0 \leq t < \frac{1}{2}$$

$$\left. \begin{aligned} \psi_{2,0}^2 &= \frac{2\sqrt{2}}{\sqrt{\pi}} \\ \psi_{2,1}^2 &= \frac{2\sqrt{2}}{\sqrt{\pi}} (8t - 6) \\ \psi_{2,2}^2 &= \frac{2\sqrt{2}}{\sqrt{\pi}} (64t^2 - 96t + 35) \\ \psi_{2,3}^2 &= \frac{2\sqrt{2}}{\sqrt{\pi}} (512t^3 - 1152t^2 + 848t - 204) \\ \psi_{2,4}^2 &= \frac{2\sqrt{2}}{\sqrt{\pi}} (4096t^4 - 12288t^3 + 13632t^2 - 662t + 1189) \\ \psi_{2,5}^2 &= \frac{2\sqrt{2}}{\sqrt{\pi}} (32768t^5 - 122880t^4 + 182272t^3 - 133632t^2 + 48408t - 6930) \\ \psi_{2,6}^2 &= \frac{2\sqrt{2}}{\sqrt{\pi}} (262144t^6 - 1179648t^5 + 2191360t^4 - 2150400t^3 + 1175424t^2 - 339264t + 40391) \end{aligned} \right\} \text{where } \frac{1}{2} \leq t < 1$$

In matrix form, the powers of t can be rewritten as follows

$$T_{\Psi^2} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 \\ \frac{5}{64} & \frac{4}{64} & \frac{1}{64} & 0 & 0 & 0 & 0 \\ \frac{14}{512} & \frac{14}{512} & \frac{6}{512} & \frac{1}{512} & 0 & 0 & 0 \\ \frac{42}{4096} & \frac{48}{4096} & \frac{27}{4096} & \frac{8}{4096} & \frac{1}{4096} & 0 & 0 \\ \frac{66}{16384} & \frac{165}{32768} & \frac{55}{16384} & \frac{22}{16384} & \frac{5}{16384} & \frac{1}{32768} & 0 \\ \frac{429}{262144} & \frac{286}{131072} & \frac{429}{262144} & \frac{104}{131072} & \frac{65}{262144} & \frac{6}{131072} & \frac{1}{262144} \end{bmatrix} \quad \text{if } 0 \leq t < \frac{1}{2}$$

and

$$T_{\Psi^2} = \frac{\sqrt{\pi}}{2\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{6}{8} & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 \\ \frac{37}{64} & \frac{12}{64} & \frac{1}{64} & 0 & 0 & 0 & 0 \\ \frac{117}{256} & \frac{55}{256} & \frac{9}{256} & \frac{1}{256} & 0 & 0 & 0 \\ \frac{1514}{4096} & \frac{456}{2048} & \frac{219}{4096} & \frac{12}{2048} & \frac{1}{4096} & 0 & 0 \\ \frac{4998}{16384} & \frac{7205}{32768} & \frac{1125}{16384} & \frac{182}{16384} & \frac{15}{16384} & \frac{1}{32768} & 0 \\ \frac{67181}{262144} & \frac{27738}{131072} & \frac{21069}{262144} & \frac{2232}{131072} & \frac{545}{262144} & \frac{18}{131072} & \frac{1}{262144} \end{bmatrix} \quad \text{if } \frac{1}{2} \leq t < 1$$

Where $A = \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \\ t^4 \\ t^5 \\ t^6 \end{pmatrix}$, $B = \begin{pmatrix} \Psi_{1,0}^2 \\ \Psi_{1,1}^2 \\ \Psi_{1,2}^2 \\ \Psi_{1,3}^2 \\ \Psi_{1,4}^2 \\ \Psi_{1,5}^2 \\ \Psi_{1,6}^2 \end{pmatrix}$

6. Application of Chebyshev Wavelets Spectral Method

In this section, the Operational matrices of first and second Chebyshev wavelets of integration will be applied to solve equation (1).

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The necessary condition for $x(t)$ to extremize $J(x)$ is that it should satisfy the Euler-Lagrang equation [5].

$$\frac{\partial F}{\partial x} - \frac{d}{dx} \left(\frac{\partial F}{\partial \dot{x}} \right) = 0$$

with the given boundary conditions for all functions

$$\begin{aligned} y_1(a) &= \alpha_1 & y_1(b) &= \beta_1 \\ y_2(a) &= \alpha_2 & y_2(b) &= \beta_2 \\ &\vdots & &\vdots \\ y_n(a) &= \alpha_n & y_n(b) &= \beta_n \end{aligned}$$

The spectral method has been applied to solve variational problems using operational Chebyshev wavelets of integration.

Suppose, the variable $y^i(t)$ can be expressed approximately as

$$y^i(t) = C^T \Psi_{n,m}(t) \quad \dots(14)$$

Using equation (13), $y^i(t), \dots, y^{i-1}(t), y(t)$ can be represented as

$$\begin{aligned} y^{i-1}(t) &= C^T \int_0^t \Psi_{n,m}(t) dt + y^{i-1}(0) = C^T P \Psi_{n,m}(t) + y^{i-1}(0) \\ &\vdots \\ y(t) &= C^T P^i \Psi_{n,m}(t) + y^{i-1}(0)t^{i-2} + y^{i-2}(0)t^{i-3} + \dots + y(0) \end{aligned}$$

Some examples are given to illustrate the method

Example (1)

Consider the following variational problem

$$\text{Min } v[y] = \int_0^1 (y'^2 - y^2) dt \quad \dots(15)$$

with the boundary conditions

$$y(0) = 0, \quad y(1) = 1 \quad \dots(16)$$

the corresponding Euler Lagrange equation is

$$y'' = -y \quad \dots(17)$$

The exact solution for this problem is $y(t) = \frac{\sin t}{\sin 1}$ with the boundary conditions (16). Two cases are considered to solve this problem :

Case (1): using first Chebyshev wavelets, $\Psi_{nm} = \Psi_{n,m}^1$,

$$C^T \Psi_{n,m}^1 + C^T P_{\Psi^1}^2 \Psi_{n,m}^1 + y'(0)t + y(0) = 0 \quad \dots(18)$$

$$C^T (1 + P_{\Psi^1}^2) \Psi_{n,m}^1 + t = 0 \quad \dots(19)$$

the variable (t) in (19) can be also expressed in terms of first Chebyshev wavelets as:

$$1.1883952 t = d^T \Psi_{n,m}^1 \quad \dots(20)$$

rewrite equation (20) to be

$$C^T (1 + P_{\Psi^1}^2) \Psi_{n,m}^1 + d^T \Psi_{n,m}^1 = 0 \quad \dots(21)$$

for $M = 3$ and $k = 1$, we obtain

$$d = [0.26329694 \quad 0.18617905 \quad 0 \quad 0.78989081 \quad 0.18617905 \quad 0]^T \quad \dots(22)$$

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from equation (19), we obtain the following equations

$$\begin{aligned} \frac{67}{64} C_{10} - \frac{1}{24\sqrt{2}} C_{11} - \frac{5}{192\sqrt{2}} C_{12} &= -0.26329694 \\ \frac{1}{16\sqrt{\pi}} C_{10} + \frac{125}{128} C_{11} - \frac{1}{48} C_{12} &= -0.18617905 \\ \frac{1}{64\sqrt{2}} C_{10} + \frac{127}{128} C_{12} &= 0 \\ \frac{1}{4} C_{10} - \frac{1}{12\sqrt{2}} C_{11} - \frac{1}{6\sqrt{2}} C_{12} + \frac{67}{64} C_{20} - \frac{1}{24\sqrt{2}} C_{21} - \frac{5}{192\sqrt{2}} C_{22} &= -0.78989081 \\ \frac{1}{8\sqrt{2}} C_{10} - \frac{1}{24} C_{12} + \frac{1}{16\sqrt{2}} C_{20} + \frac{125}{128} C_{21} - \frac{1}{48} C_{22} &= -0.18617905 \\ \frac{1}{64\sqrt{2}} C_{20} + \frac{127}{128} C_{22} &= 0 \end{aligned}$$

Solving the above system to get the values of C^T

$$C = [-0.25649438 \quad -0.17897880 \quad 0.00285620 \quad -0.70668834 \quad -0.13516132 \quad 0.00786936]^T$$

Case (2): $\Psi_{nm} = \Psi_{n,m}^2$

$$C^T \Psi_{n,m}^2 + C^T P_{\Psi^2} \Psi_{n,m}^2 + y'(0)t + y(0) = 0 \quad \dots(23)$$

$$C^T (1 + P_{\Psi^2}) \Psi_{n,m}^2 + t = 0 \quad \dots(24)$$

the variable (t) in (24) can be also expressed in terms of first Chebyshev wavelets as:

$$1.18839511 t = d^T \Psi_{n,m}^2 \quad \dots(25)$$

re write equation (24) to be

$$C^T (1 + P_{\Psi^2}) \Psi_{n,m}^2 + d^T = 0 \quad \dots(26)$$

for $M = 3$ and $k = 2$, we obtain,

$$d = [0.18617905 \quad 0.093089524 \quad 0 \quad 0.55853715 \quad 0.093089524 \quad 0]^T \quad \dots(27)$$

from equation (26), we obtain the following equations

$$\begin{aligned} \frac{133}{128} C_{10} - \frac{1}{24} C_{11} + \frac{11}{384} C_{12} &= -0.18617905 \\ \frac{1}{32} C_{10} + \frac{374}{348} C_{11} + \frac{1}{96} C_{12} &= -0.093089524 \\ \frac{1}{128} C_{10} + \frac{383}{384} C_{12} &= 0 \\ \frac{1}{4} C_{10} - \frac{3}{32} C_{11} + \frac{1}{24} C_{12} + \frac{133}{128} C_{20} - \frac{1}{24} C_{21} + \frac{11}{384} C_{22} &= -0.55853715 \\ \frac{1}{16} C_{10} + \frac{1}{32} C_{20} + \frac{374}{384} C_{21} + \frac{1}{96} C_{22} &= -0.093089524 \\ \frac{1}{128} C_{20} + \frac{383}{384} C_{22} &= 0 \end{aligned}$$

Solving the above system to get the values of C^T

$$C = [-0.18281767 \quad -0.08973405 \quad 0.00143199 \quad -0.50453081 \quad -0.067701046 \quad 0.00395194]^T$$

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Table (1) numerical results of the above example.

t	$exact\ solution$	$firstChebyshev\ wavelets$	$second\ Chebyshev\ wavelets$	$AbsouteError\ exact - F.C.W $	$AbsouteError\ exact - S.C.W $
0	0	-0.0007437	-0.0001491	0.00074377	0.00014910
0.1	0.11864154	0.17897880	0.11890673	0.06033726	0.00026519
0.2	0.23609766	0.23649435	0.23638010	0.00039669	0.00028244
0.3	0.35119477	0.35073789	0.35092851	0.00045687	0.00026626
0.4	0.46278285	0.46206441	0.46255195	0.00071843	0.00023090
0.5	0.56974637	0.57047392	0.57125042	0.00072755	0.00150405
0.6	0.67101835	0.67151678	0.67269758	0.00049833	0.00167923
0.7	0.76558515	0.76582819	0.76797951	0.00024304	0.00161280
0.8	0.85250247	0.85210270	0.85362615	0.00039976	0.00112368
0.9	0.93090187	0.93034029	0.93198217	0.00056158	0.00108030
1	1	1.00054097	1.002266022	0.00054097	0.00054097
$M.E$				0.06033726	0.003643625
$L.S.$				0.003643625	0.00000952

Example (2)

Consider the following variational problem

$$\text{Min } v[y] = \int_0^1 (y'^2 + y^2) dt \quad \dots(28)$$

with the boundary conditions

$$y(0) = 0, y(1) = 1 \quad \dots(29)$$

the corresponding Euler Lagrange equation is

$$y'' = y \quad \dots(30)$$

The exact solution for this problem is $y(t) = \frac{e^t - e^{-t}}{e^1 - e^{-1}}$ with boundary conditions (29). to solve this problem by using :

Case (1): $\Psi_{nm}^1 = \Psi_{n,m}^1$

$$C^T P_{\Psi^1}^2 \Psi_{n,m}^1 + y'(0)t + y(0) - C^T \Psi_{n,m}^1 = 0 \quad \dots(31)$$

$$C^T (P_{\Psi^1}^2 - 1) \Psi_{n,m}^1 + 0.85091812 t = 0 \quad \dots(32)$$

the variable (t) in (32) can be also expressed in terms of first Chebyshev wavelets as:

$$0.85091812 t = d^T \Psi_{n,m}^1 \quad \dots(33)$$

re write equation (32) to be

$$C^T (P_{\Psi^1}^2 - 1) \Psi_{n,m}^1 + d^T \Psi_{n,m}^1 = 0 \quad \dots(34)$$

for $M = 3$ and $k = 1$, we obtain

$$d = [0.18852664 \quad 0.13330846 \quad 0 \quad 0.56557992 \quad 0.13330846 \quad 0]^T \quad \dots(35)$$

from equation (4.49), we obtain the following equations

$$\begin{aligned} -\frac{61}{64} C_{10} - \frac{1}{24\sqrt{2}} C_{11} - \frac{5}{192\sqrt{2}} C_{12} &= -0.18852664 \\ \frac{1}{16\sqrt{\pi}} C_{10} - \frac{131}{128} C_{11} - \frac{1}{48} C_{12} &= -0.13330846 \\ \frac{1}{64\sqrt{2}} C_{10} - \frac{129}{128} C_{12} &= 0 \end{aligned}$$

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$$\begin{aligned} \frac{1}{4}C_{10} - \frac{1}{12\sqrt{2}}C_{11} - \frac{1}{6\sqrt{2}}C_{12} - \frac{81}{64}C_{20} - \frac{1}{24\sqrt{2}}C_{21} - \frac{3}{192\sqrt{2}}C_{22} &= -0.56557662 \\ \frac{1}{8\sqrt{2}}C_{10} - \frac{1}{24}C_{12} + \frac{1}{16\sqrt{2}}C_{20} - \frac{131}{128}C_{21} - \frac{1}{48}C_{22} &= -0.13330846 \\ \frac{1}{64\sqrt{2}}C_{20} - \frac{129}{128}C_{22} &= 0 \end{aligned}$$

Solving the above system to get the values of C^T

$$C = [0.19347411 \quad 0.13856704 \quad 0.00212104 \quad 0.62980366 \quad 0.17393417 \quad 0.00690447]^T$$

Case 2: $\Psi_{nm} = \Psi_{n,m}^2$

$$C^T P_{\Psi^2}^2 \Psi_{n,m}^2 + y'(0)t + y(0) - C^T \Psi_{n,m}^2 = 0 \quad \dots(36)$$

$$C^T (P_{\Psi^2}^2 - 1) \Psi_{n,m}^2 + 0.85091812 t = 0 \quad \dots(37)$$

the variable (t) in (37) can be also expressed in terms of first Chebyshev

wavelets as: $0.85091812 t = d^T \Psi_{n,m}^2 \quad \dots(38)$

rewrite equation (38) to be

$$C^T (P_{\Psi^2}^2 - 1) \Psi_{n,m}^2 + d^T \Psi_{n,m}^2 = 0 \quad \dots(39)$$

for $M = 3$ and $k = 2$, we obtain

$$d = [0.13300846 \quad 0.06665423 \quad 0 \quad 0.39992525 \quad 0.06665423 \quad 0]^T \quad \dots(40)$$

from equation (39), we obtain the following equations

$$\begin{aligned} -\frac{123}{128}C_{10} - \frac{1}{24}C_{11} + \frac{11}{384}C_{12} &= -0.13300846 \\ \frac{1}{32}C_{10} - \frac{394}{348}C_{11} + \frac{1}{96}C_{12} &= -0.06665423 \\ \frac{1}{128}C_{10} - \frac{385}{384}C_{12} &= 0 \\ \frac{1}{4}C_{10} - \frac{3}{32}C_{11} + \frac{1}{24}C_{12} - \frac{123}{128}C_{20} - \frac{1}{24}C_{21} + \frac{11}{384}C_{22} &= -0.39992525 \\ \frac{1}{16}C_{10} + \frac{1}{32}C_{20} - \frac{394}{384}C_{21} + \frac{1}{96}C_{22} &= -0.06665423 \\ \frac{1}{128}C_{20} - \frac{385}{384}C_{22} &= 0 \end{aligned}$$

Solving the above system to get the values of C^T

$$C = [0.13576240 \quad 0.06911234 \quad 0.00105789 \quad 0.44114898 \quad 0.08670319 \quad 0.00343752]^T$$

Table (2) numerical results of the above example.

t	exact solution	first Chebyshev wavelets	second Chebyshev wavelets	AbsouteError exact - F.C.W	AbsouteError exact - S.C.W
0	0	0.00057584	0.00114883	0.00057548	0.00114883
0.1	0.08523370	0.08469184	0.08505153	0.00541863	0.00018217
0.2	0.17132045	0.17097404	0.17111507	0.00034641	0.00020538
0.3	0.25912184	0.29422446	0.25933944	0.03510262	0.00021760
0.4	0.34951660	0.35003704	0.34972464	0.00052044	0.00020804
0.5	0.44340944	0.44281784	0.44276674	0.00059160	0.00064270
0.6	0.54174007	0.54103704	0.54035565	0.00070303	0.00138443
0.7	0.64549262	0.64500906	0.64402083	0.00048356	0.00147179
0.8	0.75570548	0.75603258	0.75470746	0.00032710	0.00099850
0.9	0.87348169	0.87410757	0.87241554	0.00062588	0.00106615
1	1	0.99923405	0.99714506	0.00076595	0.00285494

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M.E				0.03510262	0.00285424
L.S.E				0.00123537	0.00001028

Example (3)

Consider the following variational problem

$$\text{Min } v[y] = \int_0^1 (y'^2 - y^2 - 2ty) dt \quad \dots(41)$$

with the boundary conditions

$$y(0) = 1, y(1) = 2 \quad \dots(42)$$

the corresponding Euler Lagrange equation is

$$y'' + y = -t \quad \dots(43)$$

The exact solution for this problem is $y(t) = \cos t \frac{3 - \cos 1}{\sin 1} \sin t - t$

Case1: $\Psi_{nm} = \Psi_{n,m}^1$

$$C^T P_{\Psi^1}^2 \Psi_{n,m}^1 + y'(0)t + y(0) - C^T \Psi_{n,m}^1 = 0 \quad \dots(44)$$

$$C^T (P_{\Psi^1}^2 - 1) \Psi_{n,m}^1 + 0.85091812 t = 0 \quad \dots(45)$$

the variable (t) in (45) can be also expressed in terms of first Chebyshev

$$\text{wavelets as: } 0.85091812 t = d^T \Psi_{n,m}^1 \quad \dots(46)$$

rewrite equation (45) to be

$$C^T (P_{\Psi^1}^2 - 1) \Psi_{n,m}^1 + d^T \Psi_{n,m}^1 = 0 \quad \dots(47)$$

for M=4 and k=1, we obtain

$$d = [1.33857791 \quad 0.45794418 \quad 0 \quad 0 \quad 0.829119514 \quad 0.45794418 \quad 0 \quad 0] \quad \dots(48)$$

from equation (45), we obtain the following equations

$$\frac{3}{64} C_1 - \frac{1}{24\sqrt{2}} C_2 + \frac{1}{16\sqrt{2}} C_3 - \frac{1}{48\sqrt{2}} C_4 + C_1 = -1.33857791$$

$$\frac{1}{16\sqrt{2}} C_1 - \frac{3}{128} C_2 - \frac{1}{48} C_3 - \frac{1}{128} C_4 + C_2 = -0.45794418$$

$$\frac{1}{64\sqrt{2}} C_1 - \frac{1}{96} C_2 + C_3 = 0$$

$$\frac{1}{834} C_2 - \frac{1}{384} C_4 + C_4 = 0$$

$$\frac{1}{4} C_1 - \frac{1}{12} C_2 - \frac{1}{6\sqrt{2}} C_3 - \frac{1}{24\sqrt{2}} C_4 + \frac{67}{64} C_5 - \frac{1}{24} C_6 - \frac{5}{192} C_7 - \frac{1}{48\sqrt{2}} C_8 = -0.829119514$$

$$\frac{1}{8\sqrt{2}} C_1 - \frac{1}{24} C_3 + \frac{1}{16\sqrt{2}} C_5 - \frac{125}{128} C_6 - \frac{1}{48} C_7 - \frac{1}{128} C_8 = -0.45794418$$

$$\frac{1}{24\sqrt{2}} C_5 - \frac{1}{128} C_7 + C_7 = 0$$

$$\frac{1}{384} C_6 - \frac{1}{384} C_8 + C_8 = 0$$

Solving the above system to get the values of C^T

$$C = [-1.47551694 \quad -0.40180073 \quad 0.01647392 \quad 0.00104909 \quad -2.38658711 \\ -0.59450459 \quad 0.03758405 \quad 0.00155223]$$

Case (2): $\Psi_{nm} = \Psi_{n,m}^2$

$$C^T P_{\Psi^2}^2 \Psi_{n,m}^2 + y'(0)t + y(0) - C^T \Psi_{n,m}^2 = 0 \quad \dots(49)$$

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$$C^T(P_{\Psi^2}^2 + 1)\Psi_{n,m}^2 + 1.92309270 t + 1 = -t \quad \dots(50)$$

the variable (t) in (50) can be also expressed in terms of first Chebyshev

$$\text{wavelets as: } 1.92309270 t + 1 = d^T \Psi_{n,m}^2 \quad \dots(51)$$

rewrite equation (51) to be

$$C^T(P_{\Psi^2}^2 - 1)\Psi_{n,m}^2 + d^T \Psi_{n,m}^2 = 0 \quad \dots(52)$$

for $M = 4$ and $k = 2$, we obtain

$$d = [1.08460125 \quad 0.22897208 \quad 0 \quad 0 \quad 2.00048969 \quad 0.22897207 \quad 0 \quad 0]^T \dots(53)$$

from equation (52), we obtain the following equations

$$\begin{aligned} \frac{5}{128} C_1 - \frac{1}{24} C_2 + \frac{5}{192} C_3 - \frac{1}{96} C_4 + C_1 &= -1.08460125 \\ \frac{1}{32} C_1 - \frac{10}{348} C_2 + \frac{1}{96} C_3 - \frac{10}{768} C_4 + C_2 &= -0.22897208 \\ \frac{1}{128} C_1 + \frac{255}{256} C_3 &= 0 \\ \frac{1}{384} C_2 + \frac{767}{768} C_4 &= 0 \\ \frac{1}{4} C_1 - \frac{3}{32} C_2 + \frac{1}{24} C_{12} + \frac{1}{24} C_3 - \frac{1}{32} C_4 + \frac{133}{128} C_5 - \frac{1}{24} C_6 + \frac{1}{172} C_7 - \frac{7}{384} C_8 &= -2.00048969 \\ \frac{1}{16} C_1 + \frac{1}{32} C_5 + \frac{374}{384} C_6 + \frac{1}{96} C_7 - \frac{5}{768} C_8 &= -0.22897207 \\ \frac{1}{128} C_5 + \frac{255}{256} C_7 &= 0 \\ \frac{1}{384} C_6 + \frac{767}{768} C_8 &= 0 \end{aligned}$$

Solving the above system to get the values of C^T

$$C = [-1.05210142 \quad -0.20142181 \quad 0.00825178 \quad 0.00052522 \quad -1.695137701 \quad -0.11141788 \quad 0.00157978 \quad 0.00029053]^T$$

Table (3) numerical results of the above example.

t	Exact	1st- Chebyshev Polynomials M=6	1st- Chebyshev wavelets M=3	2nd- Chebyshev wavelets M=3	1st- Chebyshev wavelets M=4	2nd- Chebyshev wavelets M=4
0	1	1	0.99832938	0.99165117	0.99914691	0.99991427
0.1	1.18682650	1.19000000	1.18836458	1.18746056	1.18602769	1.18684660
0.2	1.36079545	1.36266667	1.36161154	1.36143651	1.35994092	1.36078608
0.3	1.51916945	1.51699999	1.51807025	1.51857901	1.518315183	1.51915797
0.4	1.65936691	1.65199999	1.65774071	1.65888805	1.65857905	1.65938755
0.5	1.77898785	1.76666666	1.78062292	1.78236365	1.77816108	1.77890082
0.6	1.87583791	1.85999999	1.87315222	1.86162549	1.87588680	1.87582796
0.7	1.94795021	1.93099999	1.93676639	1.93318897	1.94796871	1.94793439
0.8	1.99360306	1.97866666	1.98134699	1.97769611	1.99366865	1.99362610
0.9	2.01134714	2.00199999	2.01340751	1.99514690	2.01138196	2.01137887

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1	2	2	2.01340751	1.98554136	2.00000001	2.00000001
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Table (4) the error for the first and second Chebyshev wavelets.

t	$AbsouteError$ $ exact - C.P $	$AbsouteError$ $ exact - F.C.W $ $M = 3$	$AbsouteError$ $ exact - F.C.W $ $M = 4$	$AbsouteError$ $ exact - S.C.W $ $M = 3$	$AbsouteError$ $ exact - S.C.W $ $M = 4$
0	0	0.00167062	0.00008531	0.00834883	0.00000857
0.1	0.0031735	0.00153808	0.00007988	0.00063406	0.00000200
0.2	0.00187122	0.00081609	0.00008545	0.00064106	0.00000094
0.3	0.00216946	0.0010992	0.00008543	0.00059044	0.00000116
0.4	0.00736692	0.0016262	0.00007879	0.00047886	0.00000206
0.5	0.01232119	0.00163507	0.00008268	0.00337585	0.00000878
0.6	0.01583801	0.00268569	0.00004889	0.01421242	0.00000090
0.7	0.01695022	0.01118382	0.00001850	0.01476124	0.00000158
0.8	0.01493840	0.01225807	0.00006359	0.01590895	0.00000210
0.9	0.00934715	0.00445311	0.00003482	0.01620024	0.00000317
1	0	0.01340751	0.00000001	0.01445864	0.00000001
$M.E$	0.01695022	0.01340751	0.00008543	0.01620024	0.00000857
$L.S.E$	0.00010730	0.00034423	0.00000004	0.00122704	0.00000000

7. Conclusion

The first and second Chebyshev wavelets operational matrices of integrations with the aid of spectral method are applied to solve variational problems. The wavelets technique allows the creation of very fast algorithms when compared to the algorithms ordinarily used (Chebyshev Polynomials).

Numerical results with comparisons are given to confirm the reliability of the proposed method for solving variational problem.

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نظرية الطيف باستخدام شيبشيف الموجية (الأولى والثانية) لحل مسائل التغير.

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الجامعة التكنولوجية / قسم العلوم التطبيقية

المستخلص:

في هذا البحث ،سوف يتم ادخال مصفوفة عمليات شيبشيف الموجية للتكامل (الاولى والثانية) للحلول التقريبية لمسائل التغير .

الفكرة الرئيسة وراء هذا النهج هو تحويل مسائل التغير الى معادلات جبرية وحلها باستخدام نظرية الطيف .

مع عدد قليل من موجات شيبشيف للحصول على نتائج مرضية. وظهرت التطبيقات للطريقة المقترحة من خلال امثلة توضيحية.