Abstract

We shall study the new relation between two the quasi-prime module and pseudo-prime module. The purpose of this paper is to study and gave the new relation between the two different prime module.

Introduction:

We shall study the new relation between two different prime modules. A module is called Quasi-prime module if \( \text{ann}_R N \) is a prime idal for every \( N \) be a submodule of a module \( M \). And \( M \) is called pseudo-prime Module if \( \text{ann}_R N \) is primary ideal for every \( N \) be submodule of \( M \) \([3],[4]\),where where \( \text{ann}_R N=\{r \in R;rx=0 \text{ for each } x \in N \} \).

The primary goal of this paper to give some condition to show That there is some relation between the Quasi-prime modules and pseudo Prime modules. In the start we show that every Quasi-prime modules is Pseudo-prime modules but the converse is not true in general unless \( M \) be cyclic. And we showing that if \( N \) is pseudo-prime submodule of \( M \) then the \( R \)-homomorphism \( \varphi^{-1}(N) \) is also pseudo-prime Submodule of \( M \). And we prove if \( N \) is a pseudo-prime submodule of \( M \) and \( \varphi \) is an epimorphism such that \( \ker \varphi \subseteq N \) then \( \varphi(N) \) is a pseudo prime submodule of \( M \).

Theorem (1):

Every Quasi-prime module is pseudo-prime module

Proof:

Let \( M \) be Quasi-prime module so \( \text{ann}_R N \) is a prime ideal For every \( N \) be a submodule of a module \( M \). But every prime Ideal is primary ideal so \( \text{ann}_R N \) is a primary ideal for every \( N \) be submodule of a module \( M \). Which mean that \( M \) is a pseudo Prime module (by definition)

But the converse of the above theorem is not true. We can show that by the following example:

Let \( M=\mathbb{Z}_8 \), \( N=\{[0],[4]\} \) is pseudo prime

Since \( \text{ann}_R N=\{4\mathbb{Z}\} \) is primary ideal since \( 4 \notin \{4\mathbb{Z}\} \)

i.e \( 2.2 \in \{4\mathbb{Z}\} \) but \( 2 \notin \{4\mathbb{Z}\} \) where \( 4 \notin \{4\mathbb{Z}\} \), but \( M \) is not Quasi-prime module since \( \text{ann}_R N=\{4\mathbb{Z}\} \) isnot prime ideal

Theorem (2):

If \( M \) be cyclic module and pseudo prime then \( M \) is Quasi-prime module.
Proof:
If \( ab \in \text{ann}_R N \) to prove \( ac \in \text{ann}_R N \) or \( bc \in \text{ann}_R N \).
Since \( M \) be cyclic so every submodule is cyclic \([1]\).
Let \( N \) be submodule of a module \( M \) and \( N = \langle x \rangle; x \in M \).
Since \( ab \in \text{ann}_R N \) so \( abN = 0 \).
\( \therefore \ abx = 0 \) if \( ax \neq 0 \) then \( (bx)^n = 0 \) (since \( \text{ann}_R N \) is Primary ideal)
So \( bx = 0 \) so \( bc \in \text{ann}_R N \).
\( \therefore \ \text{ann}_R N \) is prime ideal for each \( N \) submodule of \( M \).

Remark (1):
The \( \mathbb{Z} \)-module \( \mathbb{Z}_n \) is pseudo prime iff \( n \) is prime ideal.

Proof: Trivial.
Every non-zero submodule of pseudo prime module is pseudo prime module.

Proof:
Let \( M \) be pseudo prime module and let \( N \) be Submodule of \( M \).To prove \( N \) be pseudo prime module.
Let \( U \leq N \leq M \). Since \( M \) be pseudo prime module so \( \text{Ann}_R N \) is primary ideal. So \( N \) is pseudo prime module.

Remark (3):
It is clear that if \( M \) is pseudo prime module then \( \text{ann}_R M \) is primary ideal.

Definition (1):
Let \( N \) be a submodule of a module \( M \) then \( N \) is called pseudo prime submodule of \( M \) if \( xym \in N \) for \( x, y \in R \) and \( m \in M \) then either \( x^n m \in N \) or \( Y^n m \in N \) for some \( n \in \mathbb{Z}_+ \).[3]

Definition (2):
A proper submodule \( N \) of a module \( M \) is said to be quasi-prime submodule if whenever \( r_1, r_2 \in R, m \in M \) then either \( r_1 m \in N \) or \( r_2 m \in N \) \([2]\).

Remark (4):
Every quasi-prime submodule is pseudo prime Submodule.

Proof:
Trivial, but the converse is not true for example \( (6\mathbb{Z}) \) is pseudo prime submodule of \( \mathbb{Z} \) since \( 2 \cdot 3 \cdot 2 \in (6\mathbb{Z}) \)
But \( 2^n \cdot 2 \in 6\mathbb{Z} \) whenever \( 3^n \cdot 2 \in (6\mathbb{Z}) \). But \( (6\mathbb{Z}) \) is not quasi-prime submodule \([2]\).

Definition (3):
A proper submodule \( N \) of an \( R \)-module \( M \) is semi-prime if \( r^k x \in N \) for \( r \in R, x \in M \) and \( K \) be Positive integer, then \( r x \in N \).

Remark (4):
Every quasi-prime submodule is semi-prime submodule. [2]

Remark (5):
Every semi-prime submodule is pseudo prime submodule.

**Proof:**
If \( r^k x \in N \) then \( rx \in N \) for \( k \in \mathbb{Z}, x \in M, r \in R \) (by def. of semi-prime submodule) so \( r^{k-1} rx \in N \) so either \( r^{k-1} x \in N \) or \( rx \in N \) so \( N \) is pseudo prime submodule.

**Theorem (3):**
Let \( M \) and \( M_1 \) be two \( R \)-modules and Let \( \phi : M \rightarrow M_1 \) be an \( R \)-homomorphism then
1- if \( N \) is pseudo prime submodule of \( M_1 \) then \( \phi^{-1}(N) \) is also pseudo prime submodule of \( M \).
2- if \( N \) is a pseudo prime submodule of \( M \) and \( \phi \) is an epimorphism such that \( \text{ker} \phi \leq N \) then \( \phi(N) \) is a pseudo prime submodule of \( M_1 \).

**Proof (1):**
Let \( r_1, r_2 \in R \) and \( m \in M_1 \). If \( r_1 \cdot r_2 \cdot m \in \phi^{-1}(N) \)
So either \( r_1 \cdot \phi(m) \in N \), which is pseudo prime submodule.
So either \( r_1^n \cdot \phi(m) \in N \) or \( r_2^n \cdot \phi(m) \in N \). So \( r_1^n \cdot m \in \phi^{-1}(N) \) or \( r_2^n \cdot m \in \phi^{-1}(N) \) therefore \( \phi^{-1}(N) \) is pseudo prime submodule of \( M \).

**Proof (2):**
Let \( r_1, r_2 \in R \) and \( m_1 \in M \) such that \( r_1 \cdot r_2 \cdot m_1 \in \phi(N) \)
Hence, there exist \( n \in N \) such that \( r_1 \cdot r_2 \cdot m_1 \phi(n) \).
But \( m_1 = \phi(m) \) some \( m \in M \) so \( r_1 \cdot r_2 \cdot \phi(m) = \phi(n) \). So \( r_1 \cdot r_2 \cdot (m-n) \in \ker \phi \leq N \) so \( r_1 \cdot r_2 \cdot (m-n) = n_1 \) for some \( n_1 \in N \) so \( r_1 \cdot r_2 \cdot m_1 \phi(N) \) is pseudo prime submodule of \( M \) so that there exist \( r_1^{-1} \cdot m \phi(N) \)
Or \( r_2^{-1} \cdot m \phi(N) \), which means that \( \phi(N) \) is a pseudo prime submodule of \( M_1 \).

**References:**
ON Quasi and pseudo prime module .......... Montaha Abdul-Razak Hasan

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