Approximate Solution of linear Fredholm-Volterra Integro-Differential Equations of Two-Dimensional

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Abstract

The research aims to find approximate solutions for linear Fredholm-Volterra Integro- Differential Equations of type two dimensions. Using the two-variables of the Bernstein polynomials to find the approximate solution of mixed linear Fredholm-Volterra Integro- Differential Equations of the type two dimensions. Two examples discussed in detail.

Introduction

Two dimensional integral equations play an important role in many branches of linear non linear functional analysis, engineering mathematical physics, mechanics and heat transfer problems. Therefore, many different methods are used to obtain the solution of the linear two dimensional integral equations [1]. Several methods have been proposed for numerical solution of these equations. Traditional collocation method and radial basis functions were used by z. Avazzadeh, M. Heydari and G.B.loghmani[1]. A fast numerical solution method for tow-dimensional Fredholm integral equations of the second kind based on piecewise polynomial interpolation have been publish by Fenliang, Fu-rong lin[2]. Numerical solution volterra integral equation by spectral Galerkin method were applied in[3]. Nonlinear Fredholm integral equation by spline functions have been in[4]. Linear and nonlinear volterra integral equations Homotopy method have been in[5]. Collocation methods for volterra-fredholm integral equation in[6]. New methods are always needed to solve mixed Fredholm-Volterra Integro- Differential Equation because no single method works well for all such equations. In this paper the Bernstein polynomials are used to approximate the solution of mixed linear Fredholm-Volterra Integro-Differential Equations of two dimensions .

The mixed linear Fredholm-Volterra Integro- Differential Equation of the second kind given by [7]

$$\frac{\partial^2 u(x,y)}{\partial x^2}+f(x,y)\frac{\partial u(x,y)}{\partial x}+f(x,y)\frac{\partial u(x,y)}{\partial y}+u(x,y)$$

$$=f(x,y)+\int_0^1\int_0^1 k(x,y,t,s)u(t,s)dsdt+\int_0^x\int_0^y k(x,y,t,s)u(t,s)dsdt \quad \ldots (1)$$

$$(x,y)\in D=[0,1]\times[0,1]$$

Where $k:D\times D\times R\to R$ is a continuous linear given function, $f:D\to R$ is also continuous given function and the two-variables function u(x,y) is the unknown function.

In this paper we introduce an approximate approach to solve two- dimensional linear Fredholm-Volterra Integro- Differential Equations of the second kind given in (1) using two-variables Bernstein polynomials method.

1. Bernstein Polynomials Method with Two-Variables.

The Bernstein polynomials of two-variables degree of (n+m) can be defined[7]:

$$B_{ij}^{n+m}(x,t)=\binom{n}{i}\binom{m}{j}x^i t^j(1-x)^{n-i}(1-t)^{m-j}$$

where … (2)

$$\binom{n}{i}\binom{m}{j}=\frac{n!}{i!(n-i)!}\frac{m!}{j!(m-j)!}$$

and (n,m) are the degree of polynomials and i,j are the index of polynomials

2. Solution of Two-Dimensional Linear Fredholm-Volterra Integro- Differential Equations with Bernstein Polynomials Method

In this section, Bernstein polynomials are used to find the approximate solution for the two-dimensional linear Fredholm-Volterra Integro- Differential Equations, as follows.

Recall the Fredholm-Volterra Integro- Differential Equation of the second kind given in equation.(1).

$$\frac{\partial^2 u(x,y)}{\partial x^2}+f(x,y)\frac{\partial u(x,y)}{\partial x}+f(x,y)\frac{\partial u(x,y)}{\partial y}+u(x,y)$$

$$=f(x,y)+\int_0^1\int_0^1 k(x,y,t,s)u(t,s)dsdt+\int_0^x\int_0^y k(x,y,t,s),u(t,s)dsdt$$

Let $u(x,y)=\sum_{i=0}^{n}\sum_{j=0}^{m}P_{ij}B_{ij}^{n+m}(x,y)$ … (3) where

$$B_{ij}^{n+m}(x,y)=\sum_{k=i}^{n}\sum_{l=j}^{m}(-1)^{k-i}(-1)^{l-j}\binom{n}{k}\binom{k}{i}\binom{m}{l}\binom{l}{j}x^{k}y^{l}$$

And P_{ij} control points unknown.

Substitution of the relation in equation(3) in equation (1) we get the relation

$$\frac{\partial^2(\sum_{i=0}^{n}\sum_{j=0}^{m}P_{ij}B_{ij}^{n+m}(x,y))}{\partial x^2}+f(x,y)\frac{\partial(\sum_{i=0}^{n}\sum_{j=0}^{m}P_{ij}B_{ij}^{n+m}(x,y))}{\partial x}+f(x,y)\frac{\partial(\sum_{i=0}^{n}\sum_{j=0}^{m}P_{ij}B_{ij}^{n+m}(x,y))}{\partial y}$$

$$+(\sum_{i=0}^{n}\sum_{j=0}^{m}P_{ij}B_{ij}^{n+m}(x,y)))\quad=f(x,y)$$

$$+\int_0^1\int_0^1 k(x,y,t,s)\left(\sum_{i=0}^{n}\sum_{j=0}^{m}P_{ij}B_{ij}^{n+m}(t,s)\right)dsdt+\int_0^x\int_0^y k(x,y,t,s)\left(\sum_{i=0}^{n}\sum_{j=0}^{m}P_{ij}B_{ij}^{n+m}(t,s)\right)dsdt$$

$$=f(x,y)+\int_0^1\int_0^1 k(x,y,t,s)\left(\sum_{i=0}^{n}P_{i0}B_{i0}^{n+m}(t,s)+P_{i1}B_{i1}^{n+m}(t,s)+\cdots+P_{im}B_{im}^{n+m}(t,s)\right)dsdt$$

$$+\int_0^x\int_0^y k(x,y,t,s)\left(\sum_{i=0}^{n}P_{i0}B_{i0}^{n+m}(t,s)+P_{i1}B_{i1}^{n+m}(t,s)+\cdots+P_{im}B_{im}^{n+m}(t,s)\right)dsdt$$

$$=f(x,y)+\int_0^1\int_0^1 k(x,y,t,s)\begin{pmatrix}P_{00}B_{00}^{n+m}(t,s)+P_{10}B_{10}^{n+m}(t,s)+\cdots+P_{n0}B_{n0}^{n+m}(t,s)\\+P_{01}B_{01}^{n+m}(t,s)+P_{11}B_{11}^{n+m}(t,s)+\cdots+P_{m1}B_{m1}^{n+m}(t,s)\\\cdots+P_{0m}B_{0m}^{n+m}(t,s)+P_{1m}B_{1m}^{n+m}(t,s)+\cdots P_{nm}B_{nm}^{n+m}(t,s)\end{pmatrix}dsdt$$

$$+\int_0^x\int_0^y k(x,y,t,s)\begin{pmatrix}P_{00}B_{00}^{n+m}(t,s)+P_{10}B_{10}^{n+m}(t,s)+\cdots+P_{n0}B_{n0}^{n+m}(t,s)\\+P_{01}B_{01}^{n+m}(t,s)+P_{11}B_{11}^{n+m}(t,s)+\cdots+P_{m1}B_{m1}^{n+m}(t,s)\\\cdots+P_{0m}B_{0m}^{n+m}(t,s)+P_{1m}B_{1m}^{n+m}(t,s)+\cdots P_{nm}B_{nm}^{n+m}(t,s)\end{pmatrix}dsdt$$

…(4)

Now to find all integration in equation (4). Then in order to determine P_{ij} $\quad i=0,1,\ldots,m \quad j=0,1,\cdots,m$, we need n equations;

Now Choose $x_i, i=1,2,3,\ldots n$ and $y_j, j=1,2,3,\ldots m$ in the interval [0,1]x[0,1], which give (n) equations. Solve the (n) equations by Gauss elimination to find the values P_{ij} $\quad i=0,1,\ldots,m \quad j=0,1,\cdots,m$.

The next example is an illustration to the above method.

3. Numerical Examples:

Example(1)

Consider the following two-dimensional linear Fredholm-Volterra Integro-Differential Equation of the second kind:

$$\frac{\partial^2 u(x,y)}{\partial x^2}+x\frac{\partial u(x,y)}{\partial x}+y\frac{\partial u(x,y)}{\partial y}+u(x,y)=\frac{5}{12}x-3xy+3+\frac{1}{6}y^4x^2-\frac{3}{2}xy^3-\frac{1}{4}x^2y^3$$

$$+\frac{5}{3}y+\int_0^y\int_0^x (ys)u(s,t)dsdt+\int_0^1\int_0^1 (xs-yt)u(s,t)dsdt \quad ...(5)$$

with the exact solution $u(x,y)=x-xy+3$

we choose uniform partition with m=n=1,2,3. Approximated solution for some values of (x,y) by using Bernstein polynomials method

we obtain the approximate solution as when n=m=1 in equation (3) and using the equations (4) and (5) we get

$$u(x,y)=p_{00}(1-x)(1-y)+p_{01}(1-x)y+p_{10}(1-y)x+p_{11}xy \quad ...(6)$$

Substitution of the relation in equation(6) in equation (5) and find the control points p_{ij} , i=0,1 j=0,1 are found as follows:

Find all integration in equation. Then in order to determine control points p_{ij} , i=0,1 , j=0,1 we need n equations; now choose $x_i=0,1$ and $y_j=0,1$ in the interval $[0,1]x[0,1]$, which gives (n) equations .Solve the (n) equations by Gauss elimination to find the values p_{ij}, i=0,1 , j=0,1. we obtain the approximate solution as

$$u(x,y)=3(1-x)(1-y)+3(1-x)y+4(1-y)x+3xy$$

$$=x-xy+3$$

when n=m=2 in equation (3) and using the equations (4) and (5) we get

$$u(x,y)=\sum_{i=0}^{2}\sum_{j=0}^{2}p_{ij}B_{ij}^4$$

$$u(x,y)=\sum_{i=0}^{2}\sum_{j=0}^{2}p_{ij}\binom{2}{i}\binom{2}{j}x^iy^j(1-x)^{2-i}(1-y)^{2-j}$$

$$u(x,y)=p_{00}B_0^2(x)B_0^2(y)+2p_{01}B_0^2(x)B_2^2(y)+2p_{10}B_1^2(x)B_0^2(y)$$
$$+p_{20}B_2^2(x)B_0^2(y)+p_{02}B_0^2(x)B_2^2(y)+4p_{11}B_1^2(x)B_1^2(y)$$
$$+2p_{21}B_2^2(x)B_1^2(y)+2p_{12}B_1^2(x)B_2^2(y)+p_{22}B_2^2(x)B_2^2(y)$$

$$u(x,y)=p_{00}(1-x)^2(1-y)^2+2p_{01}y(1-y)(1-x)^2+2p_{10}x(1-x)(1-y)^2$$
$$+p_{20}x^2(1-y)^2+p_{02}y^2(1-x)^2+4p_{11}xy(1-x)(1-y) \quad ...(7)$$
$$+2p_{21}x^2y(1-y)+2p_{12}x(1-x)y^2+p_{22}x^2y^2$$

Substitution of the relation in equation(7) in equation (5) and find the control points p_{ij} , i=0,1,2 j=0,1,2 are found as follows:

Find all integration in equation. Then in order to determine control points p_{ij} , i=0,1,2 j=0,1,2 we need n equations; now choose $x_i = 0,1,2$ and $y_j = 0,1,2$ in the interval $[0,1]x[0,1]$, which gives (n)equations . solve the (n) equations by Gauss elimination to find the values p_{ij}, i=0,1,2 , j=0,1,2. we obtain the approximate solution as

$$u(x,y)=3(1-x)^2(1-y)^2+2*3y(1-y)(1-x)^2+2*0.5*x(1-x)(1-y)^2$$
$$+1*x^2(1-y)^2+3y^2(1-x)^2+4*3.25xy(1-x)(1-y)$$
$$+2*3.5*x^2y(1-y)+2*3x(1-x)y^2+3x^2y^2$$
$$=x-xy+3$$

when n=m=3 in equation (3) and using the equations (4) and (5) we get

$$u(x,y)=\sum_{i=0}^{3}\sum_{j=0}^{3}p_{ij}B_{ij}^{6}$$
$$u(x,y)=\sum_{i=0}^{3}\sum_{j=0}^{3}p_{ij}\binom{3}{i}\binom{3}{j}x^i y^j(1-x)^{3-i}(1-y)^{3-j}$$
$$u(x,y)=p_{00}B_0^3(x)B_0^3(y)+3p_{01}B_0^3(x)B_1^3(y)+3p_{02}B_0^3(x)B_2^3(y)$$
$$+p_{03}B_0^3(x)B_3^3(y)+3p_{10}B_1^3(x)B_0^3(y)+9p_{11}B_1^3(x)B_1^3(y)$$
$$+9p_{12}B_1^3(x)B_2^3(y)+3p_{13}B_1^3(x)B_3^3(y)+3p_{20}B_2^3(x)B_0^3(y)$$
$$+9p_{21}B_2^3(x)B_1^3(y)+9p_{22}B_2^3(x)B_2^3(y)+3p_{23}B_2^3(x)B_3^3(y)$$
$$+p_{30}B_3^3(x)B_0^3(y)+3p_{31}B_3^3(x)B_1^3(y)+3p_{32}B_3^3(x)B_2^3(y)$$
$$+p_{33}B_3^3(x)B_3^3(y)$$
$$u(x,y)=p_{00}(1-x)^3(1-y)^3+3p_{01}y(1-y)^2(1-x)^3+3p_{02}y^2(1-y)(1-x)^3$$
$$+p_{03}y^3(1-x)^3+3p_{10}x(1-x)^2(1-y)^3+9p_{11}xy(1-x)^2(1-y)^2$$
$$+9p_{12}xy^2(1-x)^2(1-y)+3p_{13}xy^3(1-x)^2+3p_{20}x^2(1-x)(1-y)^3 \quad ..$$
$$+9p_{21}x^2y(1-y)^2(1-x)+9p_{22}x^2y^2(1-y)(1-x)+3p_{23}x^2y^3(1-x)$$
$$+p_{30}x^3(1-y)^3+3p_{31}x^3y(1-y)^2+3p_{32}x^3y^2(1-y)+p_{33}x^3y^3$$

.(8)

Substitution of the relation in equation (8) in equation (7) and find the control points p_{ij} , i=0,1,2,3 j=0,1,2,3 are found as follows:

Find all integration in equation above . Then in order to determine control points p_{ij} , i=0,1,2,3, j=0,1,2,3 we need n equations; now choose $x_i = 0,1,2,3$ and $y_j = 0,1,2,3$ in the interval $[0,1]x[0,1]$, which gives (n)equations . solve the (n) equations by Gauss elimination to find the values p_{ij}, i=0,1,2,3, j=0,1,2,3. we obtain the approximate solution as

$$u(x,y) = 3(1-x)^3(1-y)^3 + 3*3y(1-y)^2(1-x)^3 + 3*3y^2(1-y)(1-x)^3$$
$$+3y^3(1-x)^3 + 3\frac{10}{3}x(1-x)^2(1-y)^3 + 9\frac{29}{9}xy(1-x)^2(1-y)^2$$
$$+9\frac{28}{9}xy^2(1-x)^2(1-y) + 3*3xy^3(1-x)^2 + 3\frac{11}{3}x^2(1-x)(1-y)^3$$
$$+9\frac{31}{9}x^2y(1-y)^2(1-x) + 9\frac{29}{9}x^2y^2(1-y)(1-x) + 3*3x^2y^3(1-x)$$
$$+4x^3(1-y)^3 + 3\frac{11}{3}x^3y(1-y)^2 + 3\frac{10}{3}x^3y^2(1-y) + 3x^3y^3$$
$$= x - xy + 3$$

Example(2)

Consider the following two-dimensional linear Fredholm-Volterra Integro-Differential Equation of the second kind:

$$\frac{\partial u(x,y)}{\partial x} + \frac{\partial u(x,y)}{\partial y} + u(x,y) = 2x + 6y + \frac{35}{12} + 2xy^3 - \frac{5}{2}x^2y^2 - \frac{7}{2}xy^2 - x^3y + 5x^2y$$
$$+\int_0^y \int_0^x (x-s)u(s,t)\,ds\,dt + \int_0^1 \int_0^1 (yt^2 - xy^2)u(s,t)\,ds\,dt$$

with the exact solution $u(x,y) = 2x + 6y - 5$

In the same way you can find the approximate solution when n=m, for n=1,2 and using the equations (3) and (4) we get

$$u(x,y) = -5(1-x)(1-y) + 1*(1-x)y - 3(1-y)x + 3xy$$
$$= 2x + 6y - 5$$
$$u(x,y) = -5(1-x)^2(1-y)^2 - 2*2y(1-y)(1-x)^2 - 2*4x(1-x)(1-y)^2$$
$$-3x^2(1-y)^2 + 1*y^2(1-x)^2 - 4xy(1-x)(1-y)$$
$$+2*0x^2y(1-y) + 2*2x(1-x)y^2 + 3x^2y^2$$
$$= 2x + 6y - 5$$

4. Conclusion

This paper presents the use of the Bernstein polynomials method, for solving two-dimensional mixed linear Fredholm-Volterra Integro- Differential Equations of the second kind. From solving some numerical examples the following points have been identified:

1. This method can be used to solve of mixed linear Fredholm-Volterra Integro- Differential Equations.
2. It is clear that using the Bernstein polynomial basis function to approximate when the n^{th} degree of Bernstein polynomial increases the error is decreases.

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حل تقريبي لخليط معادلات فريدهوم_فولتير الخطية التكاملية التفاضلية ثنائية الابعاد

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الخلاصة:

يهدف البحث الى إيجاد الحلول التقريبية لخليط معادلة فريدهوم_فولتيرا التفاضلية التكاملية الخطية من النوع ثنائية الابعاد. باستخدام متعدد حدود برنستن ذات المتغيرين تم ايجاد الحل التقريبي لخليط معادلة فريدهوم_فولتيرا التكاملية التفاضلية الخطية من النوع ثنائية الابعاد. مثالين بحثت بالتفصيل..