Equivalence Between System of Volterra Integro-Fractional Differential Equations and Volterra Integral Equations

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Abstract

In this paper, an important theorem of equivalence between system of linear Volterra integro-fractional differential equations (LVIFDE's) with constant coefficients in Caputo derivative scene and the corresponding well-known Volterra integral equation system is demonstrate.

Key wards: system of Volterra integro-fractional differential equations of Caputo type, Volterra integral equations

1. Introduction

The theory of fractional order has become an active area of investigation due to their applications in field such as physics, technical sciences and so on one can see [1-3], also many research groups have studied and reported on fractional problems such as integral equation with fractional time [5], fractional Volterra integral equation [8], and others [4, 6].

In this work, the following system of linear Volterra integro fractional differential equations (LVIFDE's) of Caputo type with constant coefficients in multi-terms is considered

$$ {}_a^C D_t^{\sigma_{in}} u_i(t) + \sum_{j=1}^{n-1} a_{ij}\, {}_a^C D_t^{\sigma_{i(n-j)}} u_i(t) + a_{in} u_i(t) $$

$$ = f_i(t) + \lambda \int_a^t \sum_{j=1}^{m} k_{ij}(t,s)\, u_j(s) ds\,, \quad i = 1,2,\dots,m \quad \dots(1) $$

together with initial conditions:

$$ \left[D_t^{k_i} u_i(t)\right]_{t=a} = b_i^{k_i} \;;\, k_i = 0,1,\dots,\mu_i - 1 $$

where $a \le t \le b$; $a_{ij} \in \mathbb{R}, \sigma_{ij} \in \mathbb{R}^+, i = 1,2,\dots,m$ and $j = 1,2,..,n,$ with $\sigma_{in} > \sigma_{i(n-1)} > \cdots > \sigma_{i2} > \sigma_{i1}; \mu_i = \max\{m_{ir} | r = 1,2,\dots,n\}$ and $b_i^{k_i} \in \mathbb{R}$ for all $k_i = 0,1,\dots,\mu_i - 1;$ as well as $f_i: I \to \mathbb{R}$ and $k_{ij}: S \times \mathbb{R} \to \mathbb{R}$ (with

$S = \{(t,s): a \leq s \leq t \leq b\})$ denote given (continuous) functions for each $i,j = 1,2,\dots,m$, and $u_i(t)$ is the unknown functions which are the solution of system(1).

Thus, the system of LVIFDE's for one term, two terms and three terms can be formulated as follows

(i) One term is

$$ {}_a^C D_t^{\sigma_{in}} u_i(t) = f_i(t) + \sum_{j=1}^{m} \lambda \int_a^t k_{ij}(t,s) u_j(s) ds \,, i = 1,2,\dots,m \qquad \dots(2)$$

Equipped with initial conditions: $\left[D_t^{k_i} u_i(t)\right]_{t=a} = b_i^{k_i}$; $k_i = 0,1,\dots,\mu_i - 1$.

where $\sigma_{1n} > \sigma_{2n} > \cdots > \sigma_{mn} > 0; m_{in} - 1 < \sigma_{in} \leq m_{in}\,, \mu_i = m_{in} = \lceil \sigma_{in} \rceil$ for all $i = 1,2,\dots,m$.

(ii)Two terms is

$$ {}_a^C D_t^{\sigma_{in}} u_i(t) + a_{in} u_i(t) = f_i(t) + \sum_{j=1}^{m} \lambda \int_a^t k_{ij}(t,s) u_j(s) ds \qquad \dots(3)$$

together with initial conditions: $\left[D_t^{k_i} u_i(t)\right]_{t=a} = b_i^{k_i}$; $k_i = 0,1,\dots,\mu_i - 1$

where $\sigma_{1n} > \sigma_{2n} > \cdots > \sigma_{mn} > 0; m_{in} - 1 < \sigma_{in} \leq m_{in}\,, \mu_i = m_{in} = \lceil \sigma_{in} \rceil$ for all $i = 1,2,\dots,m$.

(iii) Three terms is

$$ {}_a^C D_t^{\sigma_{in}} u_i(t) + a_{in-1}\, {}_a^C D_t^{\sigma_{i(n-1)}} u_i(t) + a_{in} u_i(t) $$
$$ = f_i(t) + \sum_{j=1}^{m} \lambda \int_a^t k_{ij}(t,s)\, u_j(s) ds, \qquad i = 1,2,\dots,m \qquad \dots(4)$$

with initial conditions: $u_i^{(k_i)}(a) = b_i^{k_i} = 0$; $k_i = 0,1,2,\dots,\mu_i - 1$, for all $i = 1,2,\dots,m$.

2. Preliminaries and Notations

In this section, some notion, definitions and preliminary facts are introduced which are used throughout this paper.

Definition (1): [1]

Let u be a real valued function defined on $[a,b]$ and $\in \mathbb{R}^+$. Then the **Riemann-Liouville (R-L) fractional integral** of order α is defined to be

$$ {}_a J_t^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}\, u(s) ds \,, \qquad \alpha > 0 $$

$$ {}_a J_t^{0} u(t) = I u(t) = u(t) $$

Definition (2): [1]

Let $\alpha \geq 0$ and $m = \lceil \alpha \rceil$ (where $\lceil \alpha \rceil$ is the ceiling function of α), the **Riemann-Liouville (R-L) fractional derivative** of order α, denote by ${}_a^R D_t^\alpha$ are defined by

$$ {}_a^R D_t^\alpha u(t) = D_t^m \, {}_a J_t^{m-\alpha} u(t) $$

$$ = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dt}\right)^m \int_a^t (t-s)^{m-\alpha-1} u(s) ds, \qquad (m-1 < \alpha \leq m) \qquad \dots (5) $$

If $= m$, $m \in \mathbb{N}_0$, and $u \in C^m[a,b]$ we have

${}_a^R D_t^0 u(t) = u(t)$; ${}_a^R D_t^m u(t) = u^{(m)}(t)$.

Definition (3): [7]

Let $\alpha \in \mathbb{R}^+$and $m-1 < \alpha \leq m \; (m \in \mathbb{N})$. The special operator ${}_a^C D_t^\alpha$ defined by

$$ {}_a^C D_t^\alpha u(t) = {}_a J_t^{m-\alpha} D_t^m u(t) $$

$$ = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} \left(\frac{d}{ds}\right)^m u(s) ds \qquad \dots (6) $$

for $a \leq t \leq b$, is called the **Caputo fractional derivatives** of order α.

Lemma (1): (mixed integration and differentiation) [3]

(i) Let $\alpha \geq \beta \geq 0$, then for $u(t) \in C[a,b]$, the relation valied at every point $t \in [a,b]$: ${}_a^R D_t^\beta \, {}_a J_t^\alpha u(t) = {}_a J_t^{\alpha-\beta} u(t)$ ... (7)

In particular, where $\beta = k \in \mathbb{N}$ and $\alpha > k$, then $D_t^k \, {}_a J_t^\alpha u(t) = {}_a J_t^{\alpha-k} u(t)$

(ii)Let $\alpha \geq \beta \geq 0$,if the fractional derivative ${}_a^R D_t^\beta$, $(m-1 < \beta \leq m)$, of a function $u(t)$ is integrable,(or, if $u(t) \in C[a,b]$ and ${}_a J_t^{m-\beta} u(t) \in C^m[a,b]$),

$$ {}_a J_t^\alpha \, {}_a^R D_t^\beta u(t) = {}_a J_t^{\alpha-\beta} u(t) - \sum_{j=1}^{m} \left[{}_a^R D_t^{\beta-j} u(t) \right]_{t=a} \frac{(t-a)^{\alpha-j}}{\Gamma(1+\alpha-j)} \qquad \dots (8) $$

Remark (1):

(a) The R-L fractional integral operators commute, i.e.,

$$ {}_a J_t^\alpha \, {}_a J_t^\beta u(t) = {}_a J_t^\beta \, {}_a J_t^\alpha u(t) = {}_a J_t^{\alpha+\beta} u(t) \qquad \dots (9/a) $$

$u(t) \in C[a,b]$, $\alpha \geq 0$, $\beta \geq 0$.

(b) For $\alpha = \beta$ in equ. (7) from, lemma (1), the R-L fractional differentiation operator is a left inverse to the R-L fractional integration operator of the same order α , i.e.

$$ {}_a^R D_t^\alpha \, {}_a J_t^\alpha u(t) = u(t), \quad a \leq t \leq b \qquad \dots (9/b) $$

(c) For $\alpha \in \mathbb{R}^+, m-1 < \alpha \leq m$ .Thusforeach $j = 1,2,\dots,m$

$$ {}_a^R D_t^{\alpha-j} u(t) = \left(\frac{d}{dt}\right)^{m-j} {}_aJ_t^{m-\alpha} u(t) \qquad \ldots(9/c)$$

Theorem (1): [6]

Let $\alpha \geq 0$ and $m = \lceil \alpha \rceil$. Moreover, assume that ${}_a^R D_t^{\alpha} u$ exists and u possesses $(m-1)$ derivative at a. Then,

$${}_a^C D_t^{\alpha} u(t) = {}_a^R D_t^{\alpha}\big[u(t) - T_{m-1}[u;a]\big], \qquad (m-1 < \alpha \leq m)$$

where $T_{m-1}[u;a]$ denotes the Taylor polynomial of degree $m-1$ for the function u, centered at a.

Remark (2): [1]

Let $u \in C^m[a,b], m \in \mathbb{N}$ and $m-1 < \alpha \leq m$. Then it is valid (from Theorem (1)):

(i) ${}_a^C D_t^{\alpha} u(t) = {}_a^R D_t^{\alpha}\left(u(t) - \sum_{k=0}^{m-1} \frac{u^{(k)}(a)}{k!}(t-a)^k\right), t \geq a \qquad \ldots(10)$

(ii) ${}_a^C D_t^{\alpha} u(t) = {}_a^R D_t^{\alpha} u(t) - \sum_{k=0}^{m-1} \frac{u^{(k)}(a)}{\Gamma(k-\alpha+1)}(t-a)^{k-\alpha}, t > a \qquad \ldots(11)$

Lemma (2): [7]

Assume that $\alpha \geq 0, m = \lceil \alpha \rceil, and\ u \in C^m[a,b]$. Then

$${}_aJ_t^{\alpha}\, {}_a^C D_t^{\alpha} u(t) = u(t) - \sum_{k=0}^{m-1} \frac{u^{(k)}(a)}{k!}(t-a)^k$$

The classic $(m \in \mathbb{N})$-fold integral and differential operators of integer-order satisfy like formula;

$$D_t^m\, {}_aJ_t^m u(t) = u(t) \ ; \ {}_aJ_t^m D_t^m u(t) = u(t) - \sum_{k=0}^{m-1} \frac{u^{(k)}(a)}{k!}(t-a)^k$$

Lemma (3): [3]

The R-L fractional integration operator ${}_aJ_t^{\alpha}$ with $\alpha > 0$ is bounded from $C[a,b]$ into $C[a,b]$:

$$\| {}_aJ_t^{\alpha} u\|_{C[a,b]} \leq \frac{(b-a)^{\alpha}}{\Gamma(1+\alpha)} \|u\|_{C[a,b]}$$

Lemma (4): (new)

Let $\alpha > \beta \geq 0, m_\alpha - 1 < \alpha \leq m_\alpha$ and $m_\beta - 1 < \beta \leq m_\beta$ $(m_\alpha, m_\beta \in \mathbb{N})$ be such that $u(t) \in C^{m_\beta}[a,b]$.Then

$${}_aJ_t^{\alpha}\, {}_a^C D_t^{\beta} u(t) = {}_aJ_t^{\alpha-\beta} u(t) - \sum_{k=0}^{m_\beta-1} \frac{u^{(k)}(a)}{\Gamma(k+\alpha-\beta+1)}(t-a)^{k+\alpha-\beta}$$

Proof:

Let $\notin \mathbb{N}_0$. Using the definition of Caputo operator with equation (9/a), yields

$$
\begin{aligned}
{}_aJ_t^{\alpha}\,{}_a^CD_t^{\beta}u(t) &= {}_aJ_t^{\alpha}\left({}_aJ_t^{m_\beta-\beta}D_t^{m_\beta}u(t)\right)\\
&= {}_aJ_t^{\alpha}\,{}_aJ_t^{m_\beta-\beta}\left(D_t^{m_\beta}u(t)\right), (m_\beta-\beta>0 \;\; and\; \alpha>0)\\
&= {}_aJ_t^{\alpha-\beta+m_\beta}\left(D_t^{m_\beta}u(t)\right),\; (\alpha-\beta>0 \;\; and\; m_\beta>0)\\
&= {}_aJ_t^{\alpha-\beta}\left({}_aJ_t^{m_\beta}D_t^{m_\beta}u(t)\right)
\end{aligned}
$$

Instead of α in lemma (2), putting m_β and for all $k=0,1,\dots,m_\beta-1$ which is greater than (-1) with linearity of ${}_aJ_t^{\alpha}$, we obtain

$$
{}_aJ_t^{\alpha}\,{}_a^CD_t^{\beta}u(t) = {}_aJ_t^{\alpha-\beta}\left(u(t)-\sum_{k=0}^{m_\beta-1}\frac{u^{(k)}(a)}{k!}(t-a)^k\right)
$$

$$
= {}_aJ_t^{\alpha-\beta}u(t)-\sum_{k=0}^{m_\beta-1}\frac{u^{(k)}(a)}{\Gamma(k+\alpha-\beta+1)}(t-a)^{k+\alpha-\beta}
$$

if $=m_\beta\in\mathbb{N}$, then applying lemma (2) and using R-L properties, we obtain the lemma (4) also.

Lemma (5) :(new)

Let $\alpha\geq\beta>0\,, m_\alpha-1<\alpha\leq m_\alpha, m_\beta-1<\beta\leq m_\beta$ and $m_\alpha\,, m_\beta\in\mathbb{N}$. if $(t)\in C^{m_\alpha-1}[a,b]$, then, for each $\gamma=0,1,\dots,m_\alpha-1$ the following relation hold:

$$
{}_aJ_t^{\alpha-\beta}u(t)=\sum_{j=0}^{\gamma-1}\frac{u^{(j)}(a)}{\Gamma(\alpha-\beta+j+1)}(t-a)^{\alpha-\beta+j}
$$

$$
+\frac{1}{\Gamma(\alpha-\beta+\gamma)}\int_a^t(t-s)^{\alpha-\beta+(\gamma-1)}u^{(\gamma)}(s)ds
$$

Proof:

Using the definition of the R-L fractional integral (1), we have

$$
{}_aJ_t^{\alpha-\beta}u(t)=\frac{1}{\Gamma(\alpha-\beta)}\int_a^t(t-s)^{\alpha-\beta-1}u(s)ds
$$

Repeated γ-time integrating by parts method for integral in above equation which depends on the value of $\gamma=0,1,2,\dots,m_\alpha-1$, we have

$$\int_a^t (t-s)^{\alpha-\beta-1} u(s)ds = \sum_{j=0}^{\gamma-1} \frac{u^{(j)}(a)}{\prod_{k=1}^{j+1}(\alpha-\beta+k-1)}(t-a)^{\alpha-\beta+j}$$
$$+\frac{1}{\prod_{k=1}^{\gamma}(\alpha-\beta+k-1)}\int_a^t (t-s)^{\alpha-\beta+\gamma-1}u^{(\gamma)}(s)ds$$

with noting: $(\alpha-\beta)\prod_{k=1}^{l}(\alpha-\beta+k-1)=\Gamma(\alpha-\beta+l)$ and take $(l=\gamma \ and \ l=j+1)$, so we get

$$ {}_aJ_t^{\alpha-\beta}u(t) = \sum_{j=0}^{\gamma-1}\frac{u^{(j)}}{\Gamma(\alpha-\beta+j+1)}(t-a)^{\alpha-\beta+j}$$
$$+\frac{1}{\Gamma(\alpha-\beta+\gamma)}\int_a^t (t-s)^{\alpha-\beta+\gamma-1}u^{(\gamma)}(s)ds$$
$$=\sum_{j=0}^{\gamma-1}\frac{u^{(j)}(a)}{\Gamma(\alpha-\beta+j+1)}(t-a)^{\alpha-\beta+j}+ {}_aJ_t^{\alpha-\beta+\gamma}u^{(\gamma)}(t)$$

In particular, if the all-linear conditions are equal to zero at the lower limit, we have:

$$ {}_aJ_t^{\alpha-\beta}u(t) = {}_aJ_t^{\alpha-\beta+\gamma}u^{(\gamma)}(t) \ ; \ \gamma=0,1,2,\dots,m_\alpha-1.$$

3. Main Results

In this section we prove all the new theorems that shown the equivalence of the LSVIFDE (1.44) in various terms and the non-linear system of VIE in the sense that, if $u_i(t)\in C^r[a,b]$, $r=m_{in}$ for $\sigma_{in}\in\mathbb{N}$ and $r=m_{in}-1$ for $\sigma_{in}\notin\mathbb{N}$ satisfies one of these relations, then it also satisfy the other one.

Theorem (2):

Assume that the functions $f_i(t)$ and $k_{ij}(t,s)$ continuous real valued functions on $I=[a,b]$ and $\mathcal{S}=\{(t,s): a\le s\le t\le b\}$ respectively, then the system of LVIFDE for one-term, equ. (1) is equivalent to the system of VIE's:

$$u_i(t) = \sum_{k_i=0}^{m_{in}-1}\frac{b_i^{k_i}}{k_i!}(t-a)^{k_i}$$
$$+\frac{1}{\Gamma(\sigma_{in})}\int_a^t (t-s)^{\sigma_{in}-1}F_i\left(s,K^{(i)}U(s)\right)ds \qquad \dots(12)$$

where $F_i\left(t,K^{(i)}U(t)\right)=f_i(t)+\sum_{j=1}^{m}\left(k^{(ij)}u_j\right)(t)$ ... $(12*)$

with vector operator $K^{(i)}U(t) = \left(k^{(i1)}u_1(t), k^{(i2)}u_2(t), \dots, k^{(im)}u_m(t)\right)$ and $U(t) = \left(u_1(t), u_2(t), \dots, u_m(t)\right)$. Define the Volterra kernel operator, for all $i, j = 1,2,\dots,m$, by

$$(k^{(ij)}u_j)(t) = \lambda \int_a^t k_{ij}(t,s)u_j(s)ds$$

Let $r = m_{in}$ for $\sigma_{in} \in \mathbb{N}$ and $r = m_{in} - 1$ for $\sigma_{in} \notin \mathbb{N}$. If $u_i(t) \in C^r[a,b]$, then $u_i(t)$ satisfies the relations (1) if only if $u_i(t)$ satisfies the VIE's (12) .

Proof:

First, the necessity is proved. Let $\sigma_{in} = m_{in} \in \mathbb{N}$ and $u_i(t) \in C^{m_{in}}[a,b]$ be the solution of LVIFDE (2), by definition of kernel operator, can be formed as:

$$ {}_a^C D_t^{\sigma_{in}} u_i(t) = F_i\left(t, k^{(i1)}u_1(t), \dots, k^{(im)}u_m(t)\right),\ i = 1,2,\dots,m \qquad \dots (13)$$

Applying the operator ${}_aJ_t^{m_{in}}$ to the relation (13) and taking into account lemma (2) with initial conditions in (1), one can arrive at the system of VIE (12).

Inversely, if $u_i(t) \in C^{m_{in}}[a,b]$ satisfies the system of IE (12), then, term-by-term differentiating equation (12) using the fundamental theorem of calculus [4], yields

$$u_i^{(\ell_i)}(t) = \sum_{k_i=\ell_i}^{m_{in}-1} \frac{b_i^{k_i}}{(k_i-\ell_i)!}(t-a)^{k_i-\ell_i} + \frac{1}{(m_{in}-\ell_i-1)!}\int_a^t (t-s)^{m_{in}-\ell_i-1} F_i\left(s, K^{(i)}U(s)\right)ds \qquad \dots (14)$$

for $\ell_i = 1,2,\dots,m_{in} - 1$. Taking the limit as $t \to a$, and taking into account the continuity of the integrands in above equation , to get

$$\left[D_t^{\ell_i} u_i(t)\right]_{t=a} = u_i^{(\ell_i)}(a) = b_i^{\ell_i}\ ;\ \ell_i = 0,1,2,\dots,m_{in} - 1$$

Differentiating the equation (14) m_{in}-time ,to obtain

$$u_i^{(m_{in})}(t) = F_i\left(t, k^{(i1)}u_1(t), \dots, k^{(im)}u_m(t)\right), i = 1,2,\dots,m$$

Thus theorem (2) is proved for $\sigma_{in} \in \mathbb{N}$.

Let now $m_{in} - 1 < \sigma_{in} < m_{in}$ and $u_i(t) \in C^{m_{in}-1}[a,b], i = 1,2,\dots,m$. According to equation (10) and R-L derivative definition (1),

$$ {}_a^C D_t^{\sigma_{in}} u_i(t) = \left(\frac{d}{dt}\right)^{m_{in}} \left({}_aJ_t^{m_{in}-\sigma_{in}} \left[u_i(t) - \sum_{k_i=0}^{m_{in}-1} \frac{u_i^{(k_i)}(a)}{k_i!}(t-a)^{k_i} \right] \right) $$

By hypotheses in theorem (2), $F_i(t, K^{(i)}U) \in C[a,b]$, it follows from equation (13) that ${}_a^C D_t^{\sigma_{in}} u_i(t) \in C[a,b]$, and hence,

$$ \left({}_aJ_t^{m_{in}-\sigma_{in}} \left[u_i(t) - \sum_{k_i=0}^{m_{in}-1} \frac{u_i^{(k_i)}(a)}{k_i!}(t-a)^{k_i} \right] \right) \in C^{m_{in}}[a,b] $$

Applying lemma (1, ii) to $v_i(t) = u_i(t) - \sum_{k_i=0}^{m_{in}-1} \frac{u_i^{(k_i)}(a)}{k_i!}(t-a)^{k_i}$, and since ${}_a^R D_t^{\sigma_{in}-k_i} u_i(t) = \left({}_aJ_t^{m_{in}-\sigma_{in}} u_i(t) \right)^{(m_{in}-k_i)}$, we can obtain

$$ {}_aJ_t^{\sigma_{in}} \, {}_a^C D_t^{\sigma_{in}} u_i(t) = {}_aJ_t^{\sigma_{in}} \, {}_a^R D_t^{\sigma_{in}} \left[u_i(t) - \sum_{k_i=0}^{m_{in}-1} \frac{u_i^{(k_i)}(a)}{k_i!}(t-a)^{k_i} \right] $$

$$ = u_i(t) - \sum_{k_i=0}^{m_{in}-1} \frac{u_i^{(k_i)}(a)}{k_i!}(t-a)^{k_i} - \sum_{\ell_i=1}^{m_{in}} \frac{\left({}_aJ_t^{m_{in}-\sigma_{in}} v_i(a) \right)^{(m_{in}-\ell_i)}}{\Gamma(1+\sigma_{in}-\ell_i)} (t-a)^{\sigma_{in}-\ell_i} \quad \ldots (15) $$

Now, how to find $\left[\left(\frac{d}{dt}\right)^{m_{in}-\ell_i} {}_aJ_t^{m_{in}-\sigma_{in}} v_i(t) \right]_{t=a}$:

First, using lemma (1, i) with $k = 1$,

$$ \frac{d}{dt}\left({}_aJ_t^{m_{in}-\sigma_{in}} v_i(t) \right) = \frac{d}{dt}\left({}_aJ_t^{m_{in}-\sigma_{in}} \left[u_i(t) - \sum_{k_i=0}^{m_{in}-1} \frac{u_i^{(k_i)}(a)}{k_i!}(t-a)^{k_i} \right] \right) $$

$$ = {}_aJ_t^{m_{in}-\sigma_{in}} \left[u_i'(t) - \sum_{k_i=1}^{m_{in}-1} \frac{u_i^{(k_i)}(a)}{(k_i-1)!}(t-a)^{k_i-1} \right] $$

Repeating this process $(m_{in} - \ell_i)$-times $(\ell_i = 1, \ldots, m_{in})$, yields

$$ \left(\frac{d}{dt}\right)^{m_{in}-\ell_i} {}_aJ_t^{m_{in}-\sigma_{in}} v_i(t) $$

$$ = {}_aJ_t^{m_{in}-\sigma_{in}} \left[u_i^{(m_{in}-\ell_i)}(t) - \sum_{k_i=m_{in}-\ell_i}^{m_{in}-1} \frac{u_i^{(k_i)}(a)}{(k_i-m_{in}+\ell_i)!}(t-a)^{k_i-m_{in}+\ell_i} \right] $$

Making the change of variable $s = a + \xi(t-a)$, we can obtain, for all $\ell_i = 1,2,\ldots,m_{in}$

$$\left(\frac{d}{dt}\right)^{m_{in}-\ell_i} {}_aJ_t^{m_{in}-\sigma_{in}} v_i(t)$$
$$= \frac{(t-a)^{m_{in}-\sigma_{in}}}{\Gamma(m_{in}-\sigma_{in})} \int_0^1 (1-\xi)^{m_{in}-\sigma_{in}-1} \left[u_i^{(m_{in}-\ell_i)}(a+\xi(t-a)) - \sum_{k_i=m_{in}-\ell_i}^{m_{in}-1} \frac{u_i^{(k_i)}(a)}{(k_i-m_{in}+\ell_i)!} [\xi(t-a)]^{k_i-m_{in}+\ell_i} \right] d\xi$$

Since $\sigma_{in} < m_{in}$ and $u_i^{(m_{in}-\ell_i)}(t) \in C[a,b]$, for $\ell_i = 1,2,\dots,m_{in}$, $i = 1,2,\dots,m$ then the last relations yield $\left({}_aJ_t^{m_{in}-\sigma_{in}} v_i(a)\right)^{(m_{in}-\ell_i)} = 0$, and hence equation (15) takes the form

$${}_aJ_t^{\sigma_{in}} {}_a^CD_t^{\sigma_{in}} u_i(t) = u_i(t) - \sum_{k_i=0}^{m_{in}-1} \frac{u_i^{(k_i)}(a)}{k_i!} (t-a)^{k_i} \qquad \dots (16)$$

Let now $\sigma_{in} \notin \mathbb{N}, m_{in}-1 < \sigma_{in} \le m_{in}$ and $u_i(t) \in C^{m_{in}-1}[a,b]$.By the hypotheses of theorem (2), $F_i\left(t, K^{(i)}U(t)\right)$ continuous on $[a,b]$, and it follows from lemma (2) , ${}_aJ_t^{\sigma_{in}} F_i\left(t, K^{(i)}U(t)\right) \in C[a,b]$, that ${}_a^CD_t^{\sigma_{in}} u_i(t) \in C[a,b]; i = 1,2,\dots,m$. Applying the operator ${}_aJ_t^{\sigma_{in}}$ to either sides of (1) or (12) and use equation (16) with initial conditions in (1), it can be found that $u_i(t) \in C^{m_{in}-1}[a,b]$is a solution to the system of VIE:

$$u_i(t) = \sum_{k_i=0}^{m_{in}-1} \frac{b_i^{k_i}}{k_i!}(t-a)^{k_i} + \frac{1}{\Gamma(\sigma_{in})} \int_a^t (t-s)^{\sigma_{in}-1} F_i\left(s, k^{(i1)}u_1(s), \dots, k^{(im)}u_m(s)\right) ds$$

for all $i = 1,2,\dots,m$, and thus the necessity is proved. Conversely, let $u_i(t) \in C^{m_{in}-1}[a,b]$ be the solution to the system of VIE (12). It is necessary first; to show $u_i(t)$ satisfies the initial conditions in (1). Differentiating both sides of it and taking lemma (1, i: integer case) into account, for all $\ell_i = 1,2,\dots,m_{in}-1$ (for $\ell_i = 0$ it is clear), yields

$$u_i^{(\ell_i)}(t) = \sum_{k_i=\ell_i}^{m_{in}-1} \frac{b_i^{k_i}}{(k_i-\ell_i)!}(t-a)^{k_i-\ell_i} +$$

$$\frac{1}{\Gamma(\sigma_{in}-\ell_i)}\int_a^t (t-s)^{\sigma_{in}-\ell_i-1}F_i\left(s,K^{(i)}U(s)\right)ds \quad ...(17)$$

Making the change of variable $z=(s-a)/(t-a)$ in the integrals in (17), for all $\ell_i=1,2,\dots,m_{in}-1$, It can be found that

$$u_i^{(\ell_i)}(t)=\sum_{k_i=\ell_i}^{m_{in}-1}\frac{b_i^{k_i}}{(k_i-\ell_i)!}(t-a)^{k_i-\ell_i}+$$

$$\frac{(t-a)^{\sigma_{in}-\ell_i}}{\Gamma(\sigma_{in}-\ell_i)}\int_0^1 (1-z)^{\sigma_{in}-\ell_i-1}F_i\left(a+z(t-a),K^{(i)}U((a+z(t-a))\right)dz$$

Since $\sigma_{in}>m_{in}-1$ and F_i continuous (for each $i=1,2,\dots,m$), the integrands in these relations continuous, and taking the limit as $t\to a^+$, the following relations are obtained:

$$\left[u_i^{(\ell_i)}(a)\right]_{t=a}=u_i^{(\ell_i)}(a)=b_i^{\ell_i}\ ,\ell_i=0,1,\dots,m_{in}-1\ ,i=1,2,\dots,m$$

which is the initial conditions of equation (1). Now it's shown that $u_i(t)\in C^{m_{in}-1}[a,b]$ satisfies the equ.(13) , rewritten equation (12) :

$$u_i(t)-T_{m_{in}-1}[u_i;a]=\ {}_aJ_t^{\sigma_{in}}F_i\left(t,K^{(i)}U(t)\right)$$

Applying the operator ${}_a^RD_t^{\sigma_{in}}$ to both sides, taking theorem (1) into account and using the relation (9), the following equation is achieved

$${}_a^CD_t^{\sigma_{in}}u_i(t)=F_i\left(t,k^{(i1)}u_1(t),k^{(i2)}u_2(t),\dots,k^{(im)}u_m(t)\right)$$

$$=f_i(t)+\sum_{j=1}^{m}\lambda\int_a^t k_{ij}(t,s)u_j(s)ds\ ;\ \ i=1,2,\dots,m$$

Thus, the theorem proved for $\sigma_{in}\notin\mathbb{N}$ for all $i=1,2,..,m$.

Theorem (3):

Under the assumptions of theorem (2), the system of LVIFDE for two-term, equation (3) is equivalent to the system of VIE's:

$$u_i(t)=\sum_{k_i=0}^{m_{in}-1}\frac{b_i^{k_i}}{k_i!}(t-a)^{k_i}$$

$$+\frac{1}{\Gamma(\sigma_{in})}\int_a^t (t-s)^{\sigma_{in}-1}\,H_i\left(s,u_i(s),K^{(i)}U(s)\right)ds \quad ...\ (18)$$

where $H_i\left(t,u_i(t),K^{(i)}U(t)\right)=\bar{a}_{in}u_i(t)+F_i\left(t,K^{(i)}U(t)\right)$ and $\bar{a}_{in}=-a_{in}$.

Define, for all $,j=1,2,\dots,m$, the F_i 's and Volterra kernels operator $k^{(ij)}$ with vector operator $K^{(i)}$ as in equation (12*) theorem (2) .

Let $r=m_{in}\ for\ \sigma_{in}\in\mathbb{N}$ and $r=m_{in}-1\ for\ \sigma_{in}\notin\mathbb{N}$. If $u_i(t)\in C^r[a,b]$, then $u_i(t)$ satisfies the relation (17) if and only if it satisfies the VIE's (18).

[The **proof** is similar to the theorem (2), use same stages]

Remark (3)

In equation (1), the initial conditions are linear, which can be set equal to zero by using any linear transform, simply using:

$$v_i(t)=u_i(t)-\sum_{k_i=0}^{m_{in}-1}\frac{b_i^{k_i}}{k_i!}(t-a)^{k_i}\,,\qquad i=1,2,\dots,m$$

Thus for any linear initial conditions, $\left[D_t^{k_i}u_i(t)\right]_{t=a}=b_i^{k_i}$, the initial conditions $v_i(a)=v_i'(a)=v_i''(a)=\cdots=0$ can be obtained.

Theorem (4):

Under the assumptions of theorem (2), the system of LVIFDE for three-terms, equation (3) is equivalent to the system of VIE's:

$$u_i(t)=\frac{1}{\Gamma(\sigma_{in})}\int_a^t(t-s)^{\sigma_{in}-1}H_i(s,u_i(s),({}_a^CD_s^{\sigma_{i(n-1)}}u_i)(s),K^{(i)}U(s))\,ds$$

$$=\frac{\bar{a}_{in}}{\Gamma(\sigma_{in})}\int_a^t(t-s)^{\sigma_{in}-1}u_i(s)ds+\frac{\bar{a}_{in-1}}{\Gamma(\sigma_{in}-\sigma_{i(n-1)})}\int_a^t(t-s)^{\sigma_{in}-\sigma_{i(n-1)}-1}u_i(s)ds$$

$$+\frac{1}{\Gamma(\sigma_{in})}\int_a^t(t-s)^{\sigma_{in}-1}F_i\left(s,k^{(i1)}u_1,\dots,k^{(im)}u_m\right)ds\qquad\dots(19)$$

Where

$$H_i\left(t,u_i(t),({}_a^CD_t^{\sigma_{i(n-1)}}u_i)(t),K^{(i)}U(t)\right)$$
$$=\bar{a}_{in}u_i(t)+\bar{a}_{in-1}\,{}_a^CD_t^{\sigma_{i(n-1)}}u_i(t)+F_i\left(t,K^{(i)}U(t)\right)$$

Define the Volterra kernel and vector operators same as in theorem (2), and $\sigma_{1n}>\sigma_{2n}>\cdots>\sigma_{mn}>0$, $\sigma_{in}>\sigma_{i(n-1)}>0;\ \mu_i=m_{in}=\lceil\sigma_{in}\rceil$ with $\bar{a}_{in}=-a_{in}$ and $\bar{a}_{in-1}=-a_{in-1}$, for all $i=1,2,\dots,m$.

Proof:

For necessity, let $\sigma_{in}=m_{in}\in\mathbb{N}$ and $u_i(t)\in C^{m_{in}}[a,b]$ be the solution of system (3), which is

$$ {}_{a}^{C}D_{t}^{m_{in}} u_i(t) = H_i\left(t, u_i, \left({}_{a}^{C}D_{t}^{\sigma_{i(n-1)}} u_i\right), K^{(i)} U\right) \qquad \dots(20) $$

Applying the operator ${}_{a}J_{t}^{m_{in}}$ to the relation (20) and taking into account lemma (2) with initial conditions, to obtain

$$ u_i(t) = \sum_{k_i=0}^{m_{in}-1} \frac{u_i^{(k_i)}(a)}{k_i!}(t-a)^{k_i} + {}_{a}J_{t}^{m_{in}} H_i\left(t, u_i, \left({}_{a}^{C}D_{t}^{\sigma_{i(n-1)}} u_i\right), K^{(i)} U\right) $$

$$ = \sum_{k_i=0}^{m_{in}-1} \frac{u_i^{(k_i)}(a)}{k_i!}(t-a)^{k_i} + \bar{a}_{in-1}\, {}_{a}J_{t}^{m_{in}}\, {}_{a}^{C}D_{t}^{\sigma_{i(n-1)}} u_i(t) + {}_{a}J_{t}^{m_{in}} \bar{H}_i\left(t, u_i, K^{(i)} U\right) \qquad \dots(21) $$

where $\bar{H}_i\left(t, u_i, k^{(i)} U\right) = \bar{a}_{in} u_i(t) + F_i(t, k^{(i1)} u_1, \dots, k^{(im)} u_m).$

Since $m_{in} \geq \sigma_{i(n-1)} > 0$, using lemma (4) for middle part in (21) :

$$ {}_{a}J_{t}^{m_{in}}\, {}_{a}^{C}D_{t}^{\sigma_{i(n-1)}} u_i(t) = {}_{a}J_{t}^{m_{in}-\sigma_{i(n-1)}} u_i(t) $$

$$ - \sum_{k_i=0}^{m_{in}-1} \frac{u_i^{(k_i)}(a)}{\Gamma\left(k_i + m_{in} - \sigma_{i(n-1)} + 1\right)}(t-a)^{k_i + m_{in} - \sigma_{i(n-1)}} \qquad \dots(22) $$

Using equation (22) and result of theorem (3), to get:

$$ \begin{aligned} u_i(t) = {} & \sum_{k_i=0}^{m_{in}-1} \frac{u_i^{(k_i)}(a)}{k_i!}(t-a)^{k_i} \\ & + a_{in-1} \sum_{k_i=0}^{m_{in}-1} \frac{u_i^{(k_i)}(a)}{\Gamma\left(k + m_{in} - \sigma_{i(n-1)} + 1\right)}(t-a)^{k_i + m_{in} - \sigma_{i(n-1)}} \\ & + \frac{\bar{a}_{in}}{(m_{in}-1)!} \int_a^t (t-s)^{m_{in}-1} u_i(s)\,ds \\ & + \frac{\bar{a}_{in-1}}{\Gamma\left(m_{in} - \sigma_{i(n-1)}\right)} \int_a^t (t-s)^{m_{in} - \sigma_{i(n-1)} - 1} u_i(s)\,ds \\ & + \frac{1}{(m_{in}-1)!} \int_a^t (t-s)^{m_{in}-1} F_i\left(s, k^{(i1)} u_1(s), \dots, k^{(im)} u_m(s)\right) ds \end{aligned} $$

After using initial conditions $u_i^{(k_i)}(a) = 0$, the system of VIE is obtained.

Inversely, if $u_i \in C^{m_{in}}[a, b]$ satisfies the integral equation (19), using lemma (5), yields

$$u_i(t) = \frac{\bar{a}_{in}}{(m_{in}-1)!}\int_a^t (t-s)^{m_{in}-1} u_i(s)ds$$
$$+ \bar{a}_{in-1}\left[\sum_{k_i=0}^{\gamma-1} \frac{u_i^{(k_i)}(a)}{\Gamma(k_i+m_{in}-\sigma_{i(n-1)}+1)}(t-a)^{k_i+m_{in}-\sigma_{i(n-1)}}\right.$$
$$\left.+\frac{1}{\Gamma(m_{in}-\sigma_{i(n-1)}+\gamma)}\int_a^t (t-s)^{m_{in}-\sigma_{i(n-1)}+(\gamma-1)} u_i^{(\gamma)}(s)ds\right]$$
$$+\frac{1}{(m_{in}-1)!}\int_a^t (t-s)^{m_{in}-1} F_i\left(s, K^{(i)}U(s)\right)ds \qquad \dots(23)$$

Equation (23) is true, for each $\gamma = 0,1,2\dots, m_{in}-1$. But it can be reduced, after using the initial conditions, to

$$u_i(t) = \frac{\bar{a}_{in}}{(m_{in}-1)!}\int_a^t (t-s)^{m_{in}-1} u_i(s)ds$$
$$+\frac{\bar{a}_{in-1}}{\Gamma(m_{in}-\sigma_{i(n-1)}+\gamma)}\int_a^t (t-s)^{m_{in}-\sigma_{i(n-1)}+(\gamma-1)} u_i^{(\gamma)}(s)ds$$
$$+\frac{1}{(m_{in}-1)!}\int_a^t (t-s)^{m_{in}-1} F_i\left(s, K^{(i)}U(s)\right)ds \qquad \dots(24)$$

For first initial condition, i.e. $u_i(a)$: since the first and last integrands in (24) continue for all $m_{in} \geq 1$, $i = 1,2,\dots,m$, and for second part zero-time (that is $\gamma = 0$) . So the middle term can be written as:

$$\frac{1}{\Gamma(m_{in}-\sigma_{i(n-1)})}\int_a^t (t-s)^{m_{in}-\sigma_{i(n-1)}-1} u_i(s)ds = {}_aJ_t^{m_{in}-\sigma_{i(n-1)}} u_i(t)$$

By continuity of u_i's on $[a,b]$ and $m_{in} > \sigma_{i(n-1)}$, it can be concluded that,

$$\lim_{t\to a^+} {}_aJ_t^{m_{in}-\sigma_{i(n-1)}} u_i(t) = 0 , i = 1,2,\dots,m \qquad \dots (25)$$

Taking the limit as $t \to a^+$, taking into account the continuity of integrands with equation (25), to obtain: $u_i(a) = 0$; $i = 1,2,\dots,m$.

For second initial condition, we can take $\gamma = 1$ and differentiation (24) one-time, for all $m_{in} \geq 2$, to get

$$u_i'(t) = \frac{\bar{a}_{in}}{(m_{in}-2)!}\int_a^t (t-s)^{m_{in}-2} u_i(s)ds + \frac{\bar{a}_{in-1}}{\Gamma(m_{in}-\sigma_{i(n-1)})}\int_a^t (t-s)^{m_{in}-\sigma_{i(n-1)}-1} u_i'(s)ds + \frac{1}{(m_{in}-2)!}\int_a^t (t-s)^{m_{in}-2} F_i\left(s, K^{(i)}U(s)\right)ds$$

Taking the limit as $t \to a$, with the same reasons gets: $u_i'(a) = 0 \;, i = 1,2,\dots,m$

applying the same arguments as in the above technique for $u_i''(a), u_i'''(a), \dots$ taking $\gamma = 2,3,\dots$ respectively. So for (k_i+1)-th initial conditions take $\gamma = k_i \; (k_i = 0,1,2,\dots,m_{in}-1; i = 1,2,\dots,m)$ and (24) differentiation k_i-time,

$$u_i^{(k_i)}(t) = \frac{\bar{a}_{in}}{(m_{in}-k_i-1)!}\int_a^t (t-s)^{m_{in}-k_i-1} u_i(s)ds + \frac{\bar{a}_{in-1}}{\Gamma(m_{in}-\sigma_{i(n-1)})}\int_a^t (t-s)^{m_{in}-\sigma_{i(n-1)}-1} u_i^{(k_i)}(s)ds + \frac{1}{(m_{in}-k_i-1)!}\int_a^t (t-s)^{m_{in}-k_i-1} F_i(s, K^{(i)}U(s))ds \quad \dots (26)$$

Taking the limit as $t \to a$, and the continuity of integrands $u_i^{(k_i)}$; to obtain
$u_i^{(k_i)}(a) = 0 \;\; ; \; k_i = 0,1,\dots,m_{in}-1 \;\; ; i = 1,2,\dots,m$

Putting $k_i = m_{in}-1$ and differentiating one more time of it about t , for middle term using by parts method , to get

$$\frac{d}{dt}u_i^{(m_{in}-1)}(t) = \frac{d}{dt}\left\{\bar{a}_{in}\int_a^t u_i(s)ds + \frac{\bar{a}_{in-1}}{\Gamma(m_{in}-\sigma_{i(n-1)})} * \int_a^t (t-s)^{m_{in}-\sigma_{i(n-1)}-1} u_i^{(m_{in}-1)}(s)ds + \int_a^t F_i\left(s, K^{(i)}U(s)\right)ds\right\}$$

Using fundamental theorem of calculus with initial conditions, to have

$$u_i^{(m_{in})}(t) = \bar{a}_{in}u_i(t) + \bar{a}_{in-1}\, {}_aJ_t^{m_{in}-\sigma_{i(n-1)}} u_i^{(m_{in})}(t) + F_i\left(t, k^{(i)}U(t)\right)$$
$$= \bar{a}_{in}u_i(t) + \bar{a}_{in-1}\, {}_a^CD_t^{\sigma_{i(n-1)}} u_i(t) + F_i\left(t, k^{(i)}U(t)\right) = H_i(t, u_i, ({}_a^CD_t^{\sigma_{i(n-1)}}u_i), k^{(i)}U)$$

Thus, the theorem (4) is true for each $\sigma_{in} \in \mathbb{N}$; $i = 1,2,\dots,m$.

Now let $\sigma_{in} \notin \mathbb{N}$, $m_{in} - 1 < \sigma_{in} < m_{in}$ and $u_i(t) \in C^{m_{in}-1}[a,b]$, as well as ${}_a^R D_t^{\sigma_{i\ell}}, \ell = n, n-1$, of functions $u_i(t)$ are integrable $(i = 1,2,\dots,m)$.

Applying equation (10) with lemma (1, ii) to the following equation:

$v_i(t) = u_i(t) - \sum_{k_i=0}^{m_{in}-1} \frac{u_i^{(k_i)}(a)}{k_i!}(t-a)^{k_i}$, then the result becomes

$$ {}_aJ_t^{\sigma_{in}}\, {}_a^C D_t^{\sigma_{in}} u_i(t) = {}_aJ_t^{\sigma_{in}}\, {}_a^R D_t^{\sigma_{in}} \left[u_i(t) - \sum_{k_i=0}^{m_{in}-1} \frac{u_i^{(k_i)}(a)}{k_i!}(t-a)^{k_i} \right] $$

$$ = u_i(t) - \sum_{k_i=0}^{m_{in}-1} \frac{u_i^{(k_i)}(a)}{k_i!}(t-a)^{k_i} - \sum_{\ell_i=1}^{m_{in}} \frac{\left({}_aJ_t^{m_{in}-\sigma_{in}} v_i(a) \right)^{(m_{in}-\ell_i)}}{\Gamma(1+\sigma_{in}-\ell_i)} (t-a)^{\sigma_{in}-\ell_i} $$

As in stages before in this theorem, we have $\left({}_aJ_t^{m_{in}-\sigma_{in}} v_i(a) \right)^{(m_{in}-\ell_i)} = 0$, for all $\ell_i = 1,2,\dots,m_{in}$, $i = 1,2,\dots,m$; Thus

$$ {}_aJ_t^{\sigma_{in}}\, {}_a^C D_t^{\sigma_{in}} u_i(t) = u_i(t) - \sum_{k_i=0}^{m_{in}-1} \frac{u_i^{(k_i)}(a)}{k_i!}(t-a)^{k_i} \qquad \dots(27) $$

also, take into account $\sigma_{in} > \sigma_{i(n-1)} \geq 0$ with using lemma (4), the following relation can obtained:

$$ {}_aJ_t^{\sigma_{in}}\, {}_a^C D_t^{\sigma_{i(n-1)}} u_i(t) = {}_aJ_t^{\sigma_{in}-\sigma_{i(n-1)}} u_i(t) - \sum_{k_i=0}^{m_{in}-1} \frac{u_i^{(k_i)}(a)}{\Gamma(1+k_i+\sigma_{in}-\sigma_{i(n-1)})}(t-a)^{k_i+\sigma_{in}-\sigma_{i(n-1)}} \qquad \dots(28) $$

Rewrite equation (4) as follows:

$$ {}_a^C D_t^{\sigma_{in}} u_i(t) + a_{in-1}\, {}_a^C D_t^{\sigma_{i(n-1)}} u_i(t) = \overline{H}_i\left(t, u_i(t), K^{(i)}U(t)\right) \qquad \dots(29) $$

where $\qquad \overline{H}_i\left(t, u_i, k^{(i)}U\right) = \bar{a}_{in} u_i(t) + F_i\left(t, k^{(i)}U(t)\right)$

Since $\overline{H}_i\left(t, u_i, K^{(i)}U\right) \in C[a,b]$, by lemma (3), ${}_aJ_t^{\sigma_{in}} \overline{H}_i\left(t, u_i, K^{(i)}U\right) \in C[a,b]$. Applying the operator ${}_aJ_t^{\sigma_{in}}$ to both sides (29) with using (27) and (28), it can be seen that $u_i(t) \in C^{m_{in}-1}[a,b]$ is the solution to the integral equation:

$$ u_i(t) = \sum_{k_i=0}^{m_{in}-1} \frac{u_i^{(k_i)}(a)}{k_i!}(t-a)^{k_i} + a_{in-1} \sum_{k_i=0}^{m_{in}-1} \frac{u_i^{(k_i)}(a)}{\Gamma(1+k_i+\sigma_{in}-\sigma_{i(n-1)})}(t-a)^{k_i+\sigma_{in}-\sigma_{i(n-1)}} $$

$$+\frac{\bar{a}_{in}}{\Gamma(\sigma_{in})}\int_a^t (t-s)^{\sigma_{in}-1}u_i(s)ds+\frac{\bar{a}_{in-1}}{\Gamma(\sigma_{in}-\sigma_{i(n-1)})}\int_a^t (t-s)^{\sigma_{in}-\sigma_{i(n-1)}-1}u_i(s)ds$$

$$+\frac{1}{\Gamma(\sigma_{in})}\int_a^t (t-s)^{\sigma_{in}-1}F_i\left(s,k^{(i1)}u_1,\dots,k^{(im)}u_m\right)ds$$

For all $=1,2,\dots,m$, after using the initial conditions in (4) the equation (19) is obtained. Thus, the necessity is proved.

Conversely, let $u_i(t)\in C^{m_{in}-1}[a,b]$ be the solution to the system of VIE (19). First to show that $u_i(t)$ satisfies the initial conditions in (4), differentiating both sides of it and taking lemma (2, integer case) into account, for all $\ell_i=1,2,\dots,m_{in}-1$, with lemma (5) , obtain

$$u_i^{(\ell_i)}(t)=\frac{\bar{a}_{in}}{\Gamma(\sigma_{in}-\ell_i)}\int_a^t (t-s)^{\sigma_{in}-\ell_i-1}u_i(s)ds$$

$$+\frac{\bar{a}_{in-1}}{\Gamma(\sigma_{in}-\sigma_{i(n-1)})}\int_a^t (t-s)^{\sigma_{in}-\sigma_{i(n-1)}-1}u_i^{(\ell_i)}(s)ds$$

$$+\frac{1}{\Gamma(\sigma_{in}-\ell_i)}\int_a^t (t-s)^{\sigma_{in}-\ell_i-1}F_i\left(s,k^{(i1)}u_1,\dots,k^{(im)}u_m\right)ds$$

$$=\bar{a}_{in}\;{}_aJ_t^{\sigma_{in}-\ell_i}u_i(t)+\bar{a}_{in-1}\;{}_aJ_t^{\sigma_{in}-\sigma_{i(n-1)}}u_i^{(\ell_i)}(t)+\;{}_aJ_t^{\sigma_{in}-\ell_i}F_i\left(t,K^{(i)}U(t)\right)$$

Since $\sigma_{in}>\ell_i$ (for all $\ell_i=1,2,\dots,m_{in}-1$) ,$\sigma_{in}>\sigma_{i(n-1)}$ and F_i is continuous, $i=1,2,\dots,m$. Taking the limit as $t\to a^+$, to obtain the relations

$$u_i^{(\ell_i)}(a)=0\;;\quad \ell_i=0,1,2,\dots,m_{in}-1, i=1,2,\dots,m$$

which are the initial conditions of equation (4).

Now we show that $u_i(t)\in C^{m_{in}-1}[a,b]$ satisfies (19). Using equation (28) to rewrite equation (19) as

$$u_i(t)-T_{m_{in}-1}[u_i;a]$$

$$=\bar{a}_{in}\;{}_aJ_t^{\sigma_{in}}u_i(t)+\bar{a}_{in-1}\;{}_aJ_t^{\sigma_{in}}\,{}_a^CD_t^{\sigma_{i(n-1)}}u_i(t)$$

$$+\;{}_aJ_t^{\sigma_{in}}F_i\left(t,K^{(i)}U(t)\right)$$

Appling the operator ${}_a^RD_t^{\sigma_{in}}$to both sides, taking theorem (1) into account and using the relation (9/b), following equation is obtained

$${}_a^CD_t^{\sigma_{in}}u_i(t)+a_{in-1}\,{}_a^CD_t^{\sigma_{i(n-1)}}u_i(t)+a_{in}u_i(t)=F_i\left(t,k^{(i1)}u_1,\dots,k^{(im)}u_m\right)$$

Thus, the theorem proved for $\sigma_{in}\notin\mathbb{N}$ for all $i=1,2,\dots,m$.

Theorem (5):

Under the assumptions of theorem (2), the system of LVIFDE (1) for $n(>3)$-term, with zeros initial conditions

$u_i^{(k_i)}(a) = b_i^{k_i} = 0\ ;\ k_i = 0,1,\dots,\mu_i - 1, i = 1,2,\dots,m$

Is equivalent to the system of VIE's:

$$u_i(t) = \frac{1}{\Gamma(\sigma_{in})}\int_a^t (t-s)^{\sigma_{in}-1} H_i(s, u_i, {}_a^C D_s^{\sigma_{i(n-1)}} u_i, \dots, {}_a^C D_s^{\sigma_{i1}} u_i, K^{(i)}\, U(s))\, ds$$

$$= \frac{\bar{a}_{in}}{\Gamma(\sigma_{in})}\int_a^t (t-s)^{\sigma_{in}-1} u_i(s) ds$$

$$+ \sum_{j=1}^{n-1} \frac{\bar{a}_{ij}}{\Gamma(\sigma_{in}-\sigma_{ij})}\int_a^t (t-s)^{\sigma_{in}-\sigma_{ij}-1} u_i(s) ds$$

$$+ \frac{1}{\Gamma(\sigma_{in})}\int_a^t (t-s)^{\sigma_{in}-1} F_i\left(s, k^{(i1)} u_1, \dots, k^{(im)} u_m\right) ds \qquad \dots(30)$$

where

$$H_i\left(t, u_i, {}_a^C D_t^{\sigma_{i(n-1)}} u_i, \dots, {}_a^C D_t^{\sigma_{i1}} u_i, K^{(i)} U\right)$$

$$= \bar{a}_{in} u_i(t) + \sum_{j=1}^{n-1} \bar{a}_{ij}\, {}_a^C D_t^{\sigma_{i(n-j)}} u_i(t) + F_i\left(t, K^{(i)} U(t)\right)$$

Define the Volterra kernel operator, vector operator and F_i's are same as in the theorem (2), and $\sigma_{1n} > \sigma_{2n} > \cdots > \sigma_{mn};\ \sigma_{in} > \sigma_{i(n-1)} > \cdots > \sigma_{i1} > 0$ with $\bar{a}_{i\ell} = -a_{i\ell}$ for all $i = 1,2,\dots,m\ ,\ell = n, n-1,\dots,1.$

4. Discussion

Some actual problems are discussed that have their mathematical representation appear directly in terms of linear fractional integro-differential equation of Volterra type. Other problems, whose direct representation is in terms of integral equations, can be reduced to the system of linear Volterra integro fractional differential equations.

All the theorems that shown the equivalence of the LSVIFDE in various terms and the non-linear system of VIE were proved in the sense that, if $u_i(t) \in C^r[a,b]$, $r = m_{in}$ for $\sigma_{in} \in \mathbb{N}$ and $r = m_{in} - 1$ for $\sigma_{in} \notin \mathbb{N}$ satisfies one of these relations, then it also satisfy the other one. For the concept of fractional derivatives, Caputo's definition was adopted which is a modification of the Riemann –Liouville definition.

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