On (m,n) - Separative

EMBEDDING THEOREMS ON SEMIGROUPS

Hatem A. Amin
Al-Mustansiryah University
College of Basic Education

Introduction

In this püper we investigate some embedding problems on semigroups. In Section 2 we give necessary and sufficicient conditions for semigroups to be embeddable in right groups. We give a criterion for semigroups with zero element to be embeddable in semigroups having a zero element 0 and being unions of disjoint subgroup G_α, αeY, such that G_α G_β=0if α≠β. in Section 3 we investigate the enihedd.iag of commutative semigroups into groups, in particular, into torsion-free groups. For this purpose we define (m,n)-separativity of semigroups. In Section- 4 (m,n)-separative semigroups will be studied.

For all notationa and notione which are not defined in this paper, we refer to [1].

2. Embedding into completely regular semigroups

DBFIN IT ION 1. A semigroups is said to be a right group if it contained no proper left ideals and is right calculative. (see [3]. P.310).

For the next two theorems the following lemmas are needed ([2], p.39).

LEMMA 1. A right group is the union of a set of isomorphic disjoint groups. If e and f are distinct idem potents a right group S, then the mapping $x \rightarrow xf$ ($x \in S_e$) is an isomorphism of the group S_e upon the group S_f (see [4]. P.258).

LEMMA 2. A semigroups is a right group if and only if it is a union of disjoint subgroups such that the set of identity elements of the subgroups is a right zero subsemigroup. (See (4).p. 259)

THEREM 1. A semigroup can be embedded, in a right group if and only if it is the union of disjoint subsemigroups S_α, $\alpha \in Y$, such that each subsemigroup S_α can be embedded in a group G_α, $G_\alpha \cap G_\beta$, $=\square$ if $\alpha \neq \beta$ and for every α and β Y there exists an isomorphism of G_α onto G_β such that

(i) a $\varphi_{\alpha\beta}\,\varphi_{\beta\gamma}$ = a $\varphi_{\alpha\gamma}$ for all a $\in G_\alpha$, $\alpha_1\beta_1\gamma \in Y$;

(ii) a $\varphi_{\alpha\beta}$ b= ab for every a$\in S_\alpha$, b$\in S_\beta$, $\alpha_1\beta \in Y$;

(iii) $\varphi_{\alpha\alpha}$ is the identity mapping of G_α, $\alpha \in Y$

Proof. First we show that the conditions arc sufficient. Assume that the semigroup S satieties the conditions of the theorem and let G = $\cup\{G_\alpha : \alpha \in Y\}$ For any elements a,b of G there exist $\alpha_1\beta \in Y$ such that a$\in G_\alpha$ and b$\in C_\beta$ Let $a^o b$ = a $\varphi_{\alpha\beta}$. Thus we have defined, an operation "o" on G. We prove that G(o) is a right group. The operation is single-valued because $G_\alpha \cap G_\beta$, = $\square$ if $\alpha \neq \beta$. In order to show the associatively, let a,b,c are any elements of G. Then there exist $\alpha,\beta,\gamma\in Y$ such that $a \in G_\alpha$, $b\in G_\beta$, and $C\in G_\gamma$. Thus $ao(b^o c)a^o(b\varphi_{\beta\gamma}c)=a\varphi_{\alpha\gamma}(b\varphi_{\alpha\gamma}c)(a\varphi_{\alpha\gamma}b\varphi_{\beta\gamma})c=(a\varphi_{\alpha\beta}\varphi_{\beta\gamma}b\varphi_{\beta\gamma})c=\ (a\varphi_{\alpha\gamma}b)\varphi_{\beta\gamma}c=(a^o b)^o c$.

If $a,b\in G_\alpha$ then $a\circ b = a\varphi_{\alpha\alpha}b=ab$, that is, G(o) is a semigroup which ia the union of the disjoint subgroups $G_{\alpha\alpha}\in Y$. If $e(\in G_\alpha)$ and f $(\in G_\beta)$ are idempotent elements, then $e\circ f=e\varphi_{\alpha\beta}$ cof =ff = f. Consequently, the set of -the identity element a of the groups $G_\alpha,\alpha\in Y$, is a right zero subsemigroup. Thus G a right group by Lemma 2.

We show that S can be embedded in G. Since $S\in G$, we have only to prove that $a\circ b= ab$ for every $a,b \in S$. Let a,b any couple of elements of S. Then there exist $\alpha,\beta\in Y$ so that $a\in S_\alpha$ $(\subseteq G_\alpha)$ and $b\in S_\beta(\subseteq G_\beta)$. Thus $a\circ b=a\varphi_{\alpha\beta}b=ab$ by Condition (ii). Consequently S is a subsemigroup of G and the first part of the theorem is proved.

Conversely, assume that the semigroup S is embeddable in a right group G. By Lemma 1, G is the union of a act of isomorphic subgroups G_α, $\alpha\in Y$. Thus S is the union of some subsemigroups S_α, $\alpha\in Y$, where every S_α can be embedded in G_α. The mapping $\varphi_{\alpha\beta}:x\rightarrow xf$ $(x\in G_\alpha)$ is an isomorphism of the group G_α, upon. G_β where f is the identity of G_β (see Lemma 1). Let e,f,h be the identity elements of G_α,G_β and. G_γ, respectively, and. Let $a\in G_\alpha$, $b\in G_\beta$, $x\in S_\alpha$, $y\in S_\beta$ $\alpha,\beta,\gamma\in Y$; Then

$$a\varphi_{\alpha\beta}\varphi_{\beta\gamma} = (af)h = a(fh) = ah = a\varphi_{\alpha\gamma}$$

$x\varphi_{\alpha\beta}y = (xf)y = x(fy) = xy,= a\varphi_{\alpha\alpha} = ae = a$.

Thus the theorem is completely proved.

THEOREM 2. A semigroup having a zero element 0 can be embedded in a semigroup which has a zero element 0' and IB the union of disjoint subgroups G_α, $\alpha \in Y$ so that $G_\alpha, G_\beta = 0'$ for every $\alpha \neq \beta$, $\alpha, \beta \in Y$ if and only if It is the union of disjoint subsemigroups embedded in groups S_α, $\alpha \in Y$ such that $S_\alpha, S_\beta = 0$ for every $\alpha \neq \beta$ $\alpha, \beta \in Y$

<u>Proof</u>. Since the necessity of the coalition is trivial, we have only to show the sufficiency. Assume that the semigroup S with zero 0 is the union of disjoint subsemigroups S_α $\alpha \in Y$ such that every S_α, is embeddable in a group G_α and $S_\alpha, S_\beta = 0$ $\alpha \neq \beta$ $\alpha, \beta \in Y$ We may assume that G_α, ie generated by S_α, and so $G_\alpha \cap G_\beta, = \square$ if $\alpha \neq \beta$. let $G = \cup\{G_\alpha : \alpha \in Y\}$. We define an operation "o" on G as follows: For any elements $a \in G_\alpha$, $b \in G_\beta$, let aob = ab if $\alpha = \beta$ and aob = 0 if $\alpha \neq \beta$. It is evident, that G is a semigroup with aero element aod G ±B the union of the subgroups $G_\alpha, \alpha \in Y$. Since $S_\alpha \subseteq G_\alpha$ and $a_o b$ = ab for every a,b in S, the semigroup S is embeddable in G (S is a subsemigroup of G)

3. Embedding in groups

A commutative semigroup can be embedded in a group if and only if it is calculative. For non-connotative semigroups cancellation is an evidently necessary condition for embed ability in a group, but it is far from sufficient. The first necessary and efficient condition is due to S,Lajas (see [3],p.311)

In this auction. we investigate the embedding in groups for several c lasses of commutative semigroups and deal with the embedding of commutative semigroups into torsion-free groups.

DEPIMITION 2. lie-I, S be an arbitrary semigroup. An clement a of Swill be called a asymmetry element if xay= yax for every couple $x,y \in S$ (See [8])

LEMMA 3. The eel of all symmetry elements of a semigroup S is either empty or an ideal of S.

Proof. Let a "be a symmetry element and x,y be any elements of a semigroup S. Then, for every $s \in S$,

x(a8)y = xa(sy) = (sy)ax =s(yax)= s(xay) =

=(sx)ay= ya(sx) - y(as)x

and x(sa)y = (xa)ay = ya(xa) = (yax)s= (xay)=

=xa(ye) = (ya)ax y(sa)x.

Consequently, both as and ea belong to the set of all symmetry element a of S. Thus the theorem is proved.

THEOREM 3. A left calculative semigroup which has a symmetry element is commutative.

COROLLARY. A calculative semigroup which has a symmetry element can be embedded in a group.

Proof, Assume that -the left calculative semigroup S has a symmetry element a and. let x,y be arbitrary elements of S. By Lemma 3, as is a symmetry element of S for any a S, and (sa)(xy)

=(eax)y = x(as)y = y(as)x = (yas)x =(sa)(yx). Thus xy= yx because S is left calculative.

DEPIUITIOH 3. A semigroup S is said to be separative if $a^2= ab=b^2$ implies $a = b$ for every couple $a,b\in S$. (See [6])

THEOREM 4. A commutative separative semigroup S can

be embedded in a group if and only if some power S^n (n>1) is embeddable in a group,

Proof. If a semigroups is embeddable in a group, then every power of it is so.

Conversely, let S be a commutative separative semigroup. Assume that there exists a positive integer n so that S^n is embeddable in a group. Since there is a positive integer k such $2^k\geq n$, $S^n\supseteq S^{2k}$ and so $S^{2k}$is embeddable in a group. Consequently, it sufficed to show that S is embeddable in a group if S^2 is because by applying this particular proposition general times, we get in succession that S^{2k-1}, S^{2k-2},...,S^2 and finally, S can be embedded in a group, let S be embeddable in a group. Then S^2 is calculative. We prove that S is calculative too Let $a,x,y \in S$ such that ax = ay. Then $ax = a^2x^2=a^2yx$ and $a^2xy=a^2y^2$. The elements a^2,x^2, xy belong to S , hence $x^3= y^2= xy$ because S^2 is calculative and S is commutative, Since S is separative, we get x=y This means that S is left calculative. Similarly, S is right calculative, Thus S can be embedded in a group, and, the theorem is proved.

Before dealing with the embedding of commutative semigroups in torsion-free groups, we prove the

THEOREM 5" A esmigroup which is not a group can not be embedded in a torsion group.

Proof. Let G be a torsion group. Then, for any element a of G there exists a positive integer n so that $a^n = 1$ (the identity of G). We prove that every subsemigroup of G is left simple and. right simple. First, we show that, if K is a subsemigroup of G, then. the left idealizer of K, $Id_L = \{x \in S: xK \subseteq K)$ and the right idealizer of K equal to K. Since G is a torsion group, the identity 1 of belongs to K. Thus, for any $x \in G$, $x = x1 \in xK$ and $x = lx \in Kx$. Hence $xK \subseteq K$ $[Kx \subseteq K]$ and $x \in K$. Therefore $Id_L K = K = Id_R K$. Let S be a subsemigroup of G. Then a subsemigroup K of S is a left [right] ideal of S if and only if $Id_L K \supseteq S$ $[Id_R K \supseteq S]$. Thus if K is a subsemigroup of S so that K is a left [right] ideal of S, then K = S because $S \subseteq Id_L K = K$ $[S \subseteq Id_R K = K)$. Consequently S has no proper 1eft and right ideals, whence S is a subgroup of G. Thus if a semigroup S can be embedded in a torsion group G, then S is a group. The theorem is proved.

DEPIRTTTOH 4. Let m and n be fixed positive integer so that m n. A semigroup S will be called (m,n)-separative if $a^m b^n = a^n b^m$ implies a =b for all $a,b \in S$. (see [9])

THEOREM 6. A commutative semigroup can be embedded In a t oral on-free group if and only if it is a calculative (m,n)-separat ive semigroup for all positive integer m>n.

Proof. Assume that the commutative semigroup S can be embedded in a torsion-free group G. We may assume that G is generated by S. Then G is a commutative group. let m and n be positive integers so that m>n and $a,b\in S$ with the assumption a^mb^n a^nb^m. We show that a =b. Since S is cancellative, $a^{m-n}=b^{m-n}$. Since a,b are elements of G, there exists an element x of G such that ax=b. Thus, $b^{m-n}=(ax)^{m-n}=x^{m-n} = b^{m-n}x^{m-n}$ because G is Abelian. It followed that x^{m-n} is the identity of G. Hence a=b.

Conversely, assume that a commutative semigroup S is cancellative and, for every positive integer m and n, $a^mb^n =a^n b^m$ $(a,b\in S)$ implies a=b. Since S is commutative and calculative, it is embeddable in a group G. By making use the usual construction of G [$G=SxS/\sigma$, where $(a,b)\sigma(c,d)$ iff ad=cb; $a,b,c,d\in S$), we show that G is torsion-free. Assume $(a,b)^m= (c,c)$ $(c\in S)$ for the element (a,b) of G and for some positive integer $m\geq 2$. Then $(a^m,b^m)=(c,c)$, that is, $a^mc=cb^m$ Since S is commutative, $a^mc = b^mc$. Since S is cancellative, $a^m=b^m$. Thus for any couple n>m of positive integers, $b^{n-m}a^{n-m}= b^{n-m}b^ma^{n-m}$, that is, $b^{n-m}a^n$ whence a=b by the assumption for S. Consequently, (a,b) =(c,c), $c\in S$ and so G is a torsion-free group.

4- On (m.n)-separaive semigroups

Theorem 5 showa that (m,n)-separativity is a useful condition for embedding in torsion-free groups. In, this section we inveetigate the (m,n)-separative sernigroups.

THBORBM 7. If an (m,n)-separative semigroup S contains an idempotent e, then e is the identity of S.

Proof. Let S be an (m,n)-separative semigroup, m>n, and e an idempotent of S. Then, for every $x\in S,(xe)^n(ex)^m=(xe)^{n-1}\ xex(ex)^{m-1} = (xe)^{m-1}xex(ex)^{n-1} = (xe)^m(ex)^n$

which implies xe = ex . Thus

$(xe)^n x^m = x^n e^n x^m = x^n e^m x^m$

whence xe = x. Similarly, ex » x. Thus e is the identity element of S.

THEOREK 8. If a semigroup S is a union of disjoint (n,n-2)-separative subaemigroups S_α, $\alpha\in Y$, n>2, then S is separative.

Proof, First we show -that every S_α is separative. Let a and b be any elements of S satisfying $a^2= b^2= ab$. Then, for any positive integer n> 2, $a^n b^{n-2} = a^{n-1}b^{n-1}$ and $a^{n-2}b^n= a^{n-1}b^{n-1}$, is, $a^n b^{n-2} = a^{n-2}b^n$. Hence it follows that a » b « Sow let x.ye S so that $x^2=y^2=xy$. If $x\in S_\alpha$, $\alpha\in Y$, then $y\in S_\alpha$ too. Since S_α is separative as just we have proved, x =y.

THEOREM 9. Every (2, l)-separative semigroup is separative.

Proof Let S be a (2,1)-separative semigroup, and $a,b\in S$ such that $a^2= ab = b^2$. Then $a^2b =ab^2$ which implies a= b.

THEOREM 10. If a semigroup is the union of disjoint (2,1)-separative semigroups, then it is separative.

Proof 11 trivial by Theorem 8 and Theorem 9.

THEOREM 11. A (3,1)-separative semigroup is cancellative if some power of it is cancellative.

Proof, Assume that S is a (3,1) separative semigroup and there is a positive integer n so that S^n is cancellative. We may assume that n=2 as we have proved in the proof of Theorem 4. Then S^2 is cancellative. Let a, $x,y \in S$ such that ax = ay . Then $a^2xy=a^2y^2$, $a^2yx=a^2x^2$. Consequently, y^2 = xy and x^2 because S^2 is cancellative. Thus $x^3y = xyxy = xy^3$. Since S is (3,1)- separative x=y. We can prove similarly that xa = ya implies x=y for any elements $a,x,y \in S$. Thus S is cancellative and the theorem is completely proved.

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حول نظريات الاعمار

لشبه الزمر من النمط (m,n)

حاتم محمد أمين عبد الله

الجامعة المستنصرية – كلية التربية الأساسية

من بحثنا هذا، درسنا بعض نظريات الاعمار ومسائلها لشبه الزمر. من جزء (2) قدمنا الشرط الضروري اللازم لاعمار شبه الزمر من الزمر اليمنى. خاصة اعمار تلك الشبه الزمر التي تحتوي على العنصر (O) [Zero element] ومن الجزء (3) درسنا امكانية اعمار شبه الزمر الابدالية من الزمر، خاصة من الزمر ذات ؟؟؟ من الرتب. ثم عرفتا شبه الزمر من النمط (m,n) وخواص اعمارها