Indirect Algorithm for Solving Variation Problems

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Abstract

In this work, an approximate indirect method to solve some variational problems is proposed in terms of shifted Legendre polynomials. The operational matrix of differentiation for shifted Legendre polynomials is first derived. Using the operational matrix of differentiation, the variational problems are reduced to the solution of system of algebric equations with unknown shifted Legendre coefficients. Numerical example illustrates the efficiency, simplicity and accuracy of the proposed method.

1. Introduction

Variational problems provide an alternative to search for analytical solutions of some problems. The idea is to formulate variational functional whose stationarity conditions lead to equations that describe the problems. The Euler-Lagrange equations obtained by applying the well known procedure in the calculus of variation, usually leads to equations that are difficult to solve.

In this work, we solve variational problems using indirect algorithm with the aid of shifted Legendre polynomials. First, some properties of shifted Legendre polynomials are given and then the operational matrix of differentiation is derived and indirect method for solving variational problems is presented.

2. Shifted Legendre Operational Matrix for Differentiation

Considering that the polynomial basis is used to describe a function to be composed of shifted Legendre polynomials, in the interval $[a,b]=[0,1]$, one can observe that,for shifted Legendre polynomials.

$$\dot{\bar{L}}_N(x)=\begin{cases}\sum\limits_{r=0}^{(N/2)-1}(8r+6)\bar{L}_{2r+1}(x) & n\ \ even\ .0\\ \sum\limits_{r=0}^{(N/2)}(8r+2)\bar{L}_{2r}(x) & n\ \ odd\end{cases}\qquad N=1,2,3,\ldots \qquad \ldots\ldots\ldots\ldots(1)$$

Difining

$$\bar{L}=\begin{bmatrix}\bar{L}_1\\ \bar{L}_2\\ \vdots\\ \bar{L}_N\end{bmatrix},\quad \overset{*}{\bar{L}}=\begin{bmatrix}\bar{L}_O\\ \bar{L}_1\\ \vdots\\ \bar{L}_{N-1}\end{bmatrix},\ \bar{L}_O=0$$

One can write $\frac{d}{dx}\bar{L} = D^1 \bar{\overset{*}{L}}$

Following (1), D_1 is a square matrix $D^1_{(N-1(\times(N-1)}$ and

$D^1_{(2i+j),J} = 2(2j-1)$, $j = 1,2,\ldots,i+1, i = 0,1,2,\ldots n$ …………………(2)

According to (2) the operational matrix $D^1_{7\times 7}$ can bewritten as:

$$D^1_{7\times 7} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 14 & 0 & 0 & 0 \\ 2 & 0 & 10 & 0 & 18 & 0 & 0 \\ 0 & 6 & 0 & 14 & 0 & 22 & 0 \\ 2 & 0 & 10 & 0 & 18 & 0 & 26 \end{pmatrix}$$

Similarly, one can derive the operational matrix D^2 as follows: Defining $\bar{L} = \left[\bar{L}_2\, \bar{L}_3, \ldots, L_N\,]^T, \bar{\overset{**}{L}} = [L_o\, L_1, \ldots, L_{N-2} \right]$

$$\frac{d^2}{dx^2}\bar{L} = D^2 \bar{\overset{\bullet\bullet}{L}}, \bar{\ddot{L}}_o = 0, \bar{\ddot{L}}_1 = 0$$

where the operational matrix $D^2_{8\times 8}$ can be obtained as follows:

$$D^2_{8\times 8} = \begin{pmatrix} 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 60 & 0 & 0 & 0 & 0 & 0 & 0 \\ 40 & 0 & 140 & 0 & 0 & 0 & 0 & 0 \\ 0 & 168 & 0 & 252 & 0 & 0 & 0 & 0 \\ 84 & 0 & 360 & 0 & 396 & 0 & 0 & 0 \\ 0 & 324 & 0 & 616 & 0 & 572 & 0 & 0 \\ 144 & 0 & 660 & 0 & 936 & 0 & 780 & 0 \\ 0 & 528 & 0 & 1092 & 0 & 1320 & 0 & 1020 \end{pmatrix}$$

In general, the elements of D^2 can be obtained with the use of the following :

$D^2_{(2i+j),i} = 4(2j-1)(i+1)(2(i+j)+1)$

$$j = 1,2,3,\ldots,i \quad , \quad i = 0,1,2,\ldots,N$$

3. Function Approximation

A function $f(x) \in L^2[0,1]$ may be expanded by the shifted Legendre polynomials series as follows:

$$f(x) = \sum_{r=0}^{\infty} a_r \bar{L}_r(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

where a_r are given by

$$a_r = \frac{< f, \bar{L}_r >}{< \bar{L}_r, \bar{L}_r >}, \qquad r = 0,1,2,\ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

In (4), $< .,. >$ denotes the inner product.

If the infinite series (3) is trancated up to term N, then it can be written as:

$$y(x) \cong \sum_{r=0}^{N} a_r \bar{L}_r(x) = A^T \bar{L}(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

where A and $\bar{L}$ are $(N+1)\times 1$ vectors given by:

$A = \left[a_o \; a_1 \; a_2 \; \ldots a_N \right]^T$ *and* $\bar{L}(x) = \left[\bar{L}_o \; \bar{L}_1 \; \ldots \bar{L}_N \right]^T$, furthermore, x^m can be defined as:

$$1 = [1\;0\;0\;0 \quad 0\;0]\bar{L}(x) = 1^T L,$$

$$x = \left[\frac{1}{2} \; \frac{1}{2} \; 0\;0\;\; 0\;0 \right]\bar{L}(x) = x^T L,$$

$$x^2 = \left[\frac{1}{3} \; \frac{1}{2} \; \frac{1}{6} \; 0\;\; 0\;0 \right]\bar{L}(x) = x^{2T} L,$$

$$x^3 = \left[\frac{1}{4} \; \frac{9}{20} \; \frac{1}{4} \; \frac{1}{20} \;\; 0\;0 \right]\bar{L}(x) = x^{3\,T} L,$$

$$x^4 = \left[\frac{1}{5} \; \frac{28}{70} \; \frac{20}{70} \; \frac{7}{70} \; \frac{1}{70} \;\; 0\;0 \right]\bar{L}(x) = x^{4\,T} L$$

Finally, differential of the function $y(x)$ defined in (5) can be obtained as:

$$\frac{d}{dx} y(x) = \frac{d}{dx} \sum_{r=0}^{N} a_r \bar{L}_r(x) = a^T D^1 \bar{L}(x)$$

$$\frac{d^2}{dx^2} y(x) = \frac{d}{dx} \sum_{r=0}^{N} a_r \dot{\bar{L}}_r(x) = a^T D^2 \bar{L}(x)$$

Furthermore, $\dfrac{d^n}{dx^n} y(x) = \dfrac{d^{m-1}}{dx^{m-1}} \sum_{r=0}^{N} a_r \dot{L}_r(x) = a^T D^m \bar{L}(x)$

4. The Shifted Legendre Indirect Method

Consider the problem of finding the extremum of the functional $x(t)$ The necessary condition for $x(t)$ to extremize is that it should satisfy the Euler-Lagrange equation

$$\frac{\partial f}{\partial x}-\frac{d}{dx}(\frac{\partial f}{\partial \dot{x}})=0 \quad \dots\dots(6)$$

with appropriate boundary conditions. However, the above differential equation can be integrated easily only for simple cases. Thus numerical and approximate methods have been developed to solve variational problems.

In this work, shifted Legendre polynomials are used to establish the indirect method for variational problems.

Suppose, the variable $x(t)$ can be expressed approximately as

$$x(t)=\sum_{i=0}^{m} c_i \bar{L}_i = c^T \bar{L}(t) \quad \dots\dots(7)$$

Differentiating equation (7) and using (1), we represent $\dot{x}(t)$ as

$$\dot{x}(t)=\sum_{i=0}^{m} c_i \dot{\bar{L}}_i = c^T D^1 \bar{L}(t) \quad \dots\dots(8)$$

$$\ddot{x}(t)=\sum_{i=0}^{m} c_i \ddot{\bar{L}}_i = c^T D_3^2 \bar{L}(t) \quad \dots\dots(9)$$

We can also express the functions 1,t, in terms of $\bar{L}(t)$ as follows

$$t \cong d^T \bar{L}(t) \text{ where } d^T=[d_0,d_1,\dots,d_m] \quad \dots\dots(10)$$

5. Example: (Harmonic Oscillator)

Consider the following harmonic oscillator

$$\min v[y]=\int_0^1 (y'^2-y^2)dx \quad \dots\dots(12)$$

Subject to the boundary conditions

$$y(0)=0\ ,\ y(1)=1 \quad \dots\dots(13)$$

The Euler-Lagrange equation of this problem can be written in the following form $y''=-y$ ……(14)

Whose solution subject to boundary conditions (13) is $y(x)=\frac{\sin(x)}{\sin 1}$ To solve eq.(14) by the proposed method, we assume $y(x)$ can be expanded in terms of the shifted Legendre polynomial of fourth order

$$y(x)=\sum_{r=0}^{4} a_r \bar{L}_r(x) = A^T \bar{L}(x) \quad \dots\dots(15)$$

where $A=[a_o\, a_1\, a_2\, a_3\, a_4]$, $\bar{L}(x)=[L_o\, L_1\, L_2\, L_3\, L_4]^T$. Differentiate eq. (15) twice to obtain $y'(x)$ and $y''(x)$ as follows:

$$y'(x)=\sum_{r=0}^{4} a_r \dot{\bar{L}}_r(x) = A^T D_1 \bar{L}(x) \quad \dots\dots(16) \text{ and}$$

$$y''(x)=\sum_{r=0}^{4} a_r \ddot{\bar{L}}_r(x) = A^T D_2 \bar{L}(x) \quad \dots\dots(17) \text{ where}$$

$$D^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 2 & 0 & 10 & 0 & 0 \\ 0 & 6 & 0 & 14 & 0 \end{pmatrix} \text{ and } D^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 12 & 0 & 0 & 0 & 0 \\ 2 & 60 & 10 & 0 & 0 \\ 40 & 6 & 140 & 0 & 0 \end{pmatrix}$$

Substituting (15) and (17) into (14) to get $A^T D^2 \bar{L}(x) = -A^T \bar{L}(x)$ or

$$(A^T D^2 + A^T)\bar{L}(x) = 0 \quad \text{...(18)}$$

with $A = [a_o a_1 a_2 a_3 a_4]$, $\bar{L}(x) = [\bar{L}_o \bar{L}_1 \bar{L}_2 \bar{L}_3 \bar{L}_4]^T$ the matrix equation (18) is applied to $n-2$ generating a linear algebric equation system with $n-2$ equation and variables. The two missing equations are obtained from the boundary conditions:

$$\left.\begin{aligned} y(0) &= A^T \bar{L}(0) = 0 \\ y(1) &= A^T \bar{L}(1) = 1 \end{aligned}\right\} \quad \text{...(19)}$$

Therefore, the algebric system of 5×5 equations is obtained to be:

$$12a_0 + 40a_4 + a_0 = 0$$
$$60a_3 + a_1 = 0$$
$$140a_4 + a_2 = 0$$
$$a_0 - a_1 + a_2 - a_3 + a_4 = 0$$
$$a_0 + a_1 + a_2 + a_3 + a_4 = 1$$

The solution of this system is

$$a_0 = 0.0108$$
$$a_1 = 0.5085$$
$$a_2 = 0.4927$$
$$a_3 = -0.0085$$
$$a_4 = -0.0035$$

By substituting the obtained coefficients in (15), the solution of (14) becomes

$$y(x) = 0.0108\bar{L}_0(x) + 0.5085\bar{L}_{\partial 1}(x) + 0.4927\bar{L}_2(x) - 0.0085\bar{L}_3 - 0.0035\bar{L}_4(x)$$

6. Conclusion

The operational matrices of shifted Legendre polynomials for differentiation D^1, D^2 are used to solve variational problems an explicit formula to find the elements of D^1 and D^2 are derived. The present indirect method reduces a variational problems into a set of algebric equation and give satisfactory results comparing with other outhers.