# Legendre Wavelets Method for Solving Boundary Value Problems

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## Abstract

Two techniques for solving $n^{th}$ order boundary value problem using continuous Legendre wavelets on the interval [0, 1] are presented. The first algorithm solves the boundary value problem BVP directly use the operational matrix of derivative of Legendre wavelets while the second algorithm converts the BVP into a system of Volterra integral equations then using the operational matrix of integration for Legendre wavelets, the system of integral equations is reduced to solve a set of linear algebraic equations, some examples are presented to illustrate the ability of the algorithms.

Keyword:- Legendre wavelets, boundary value problem.

## 1. Introduction

Wavelets theory is a relatively new and emerging area in mathematical research. As a powerful tool, wavelets have been extensively used in signal processing, numerical analysis and many other areas. Wavelets permit the accurate representation of variety of functions and operator [16]. In recent years interest in the solution of integral and differential equations, such as Fredholm, Volterra and integro- differential equations have greatly increased by wavelets method [5-9]. Many researchers have tried various problems using Legendre wavelets. System of Fredholm integral equation of the second kind has been solved using Legendre wavelets [9], a new operational matrix for Legendre wavelets and its application for solving fractional order boundary values problem has been introduced [16], the extended Legendre wavelets is achieved and it's properties by XiAO for solving differential equations[3]. S.A. Yousef presented a new technique named Legendre multi-scaling Ritz method for solving boundary value variation problems [1]. Many researchers employed Legendre wavelets to find the numerical solution for integro- differential equations [6]. In this paper we use Legendre wavelets function to approximate the solution of the boundary value problem in the interval [0,1] using two different algorithms.

## 2. Legendre Wavelets and Their Properties

### 2.1 Wavelets and Legendre wavelets [11,14]

Wavelets have found their way into many different fields of science and engineering. Wavelets constitute a family of signal functions constructed from dilation and translation of a signal function called mother wavelets. When the dilation parameters a and the translation parameters b vary continuously, we have the following family of continuous wavelets

$$\varphi_{a,b}(t) = |a|^{\frac{-1}{2}} \varphi\left(\frac{t-b}{a}\right) \qquad a, b \in R \;\; a \neq 0 \qquad \ldots(1)$$

Legendre wavelets $\varphi_{m,n}(t) = \varphi(k, \hat{n}, m, t)$ have four arguments k=1,2,...
$\hat{n}$=2n-1 , n=1,2,…,$2^{k-1}$ , m is the order of Legendre polynomials and t is the normalized time. They are defined on the interval [0,1] by:-

$$\varphi_{m,n}(t) = \begin{cases} \sqrt{m+\frac{1}{2}}\, 2^{k/2}\, L_m(2^k t - \hat{n}) & \frac{\hat{n}-1}{2^k} \leq t \leq \frac{\hat{n}+1}{2^k} \\ 0 & \text{otherwise} \end{cases} \qquad \ldots(2)$$

$L_m(t)$ are the well known Legendre polynomials of the order m, which are orthogonal to the weight function w(t)=1 and satisfy the following recurrence formula:-

$$L_0(t) = 1,\; L_1(t) = t \quad \text{and} \quad L_{m+1}(t) = \frac{2m+1}{m+1}\, t\, L_m(t) - \frac{m}{m+1}\, L_{m-1}(t) \qquad \ldots(3)$$

m=1,2,…,M-1

The set of Legendre wavelets are orthogonal set [2, 7]

### 2.2 Function Approximation:-

A function $f(t) \in L^2[0,1)$ may be expanded as:-

$$\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \varphi_{n,m}(t) = C^T \varphi(t) \qquad \ldots(4)$$

where $c_{n,m} = (f(t), \varphi_{n,m}(t))$ ...(5)

(. , .) denotes the inner product.

If the infinite series in equ.(4) are truncated to be written as:-

$$f(t) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \varphi_{n,m}(t) = C^T \varphi(t) \qquad \ldots(6)$$

where C and $\varphi(t)$ are $2^{k-1}M\times 1$ matrices given by:-

$$C=[c_{10}, c_{11}, \ldots, c_{20}, c_{21}, \ldots, c_{2M-1}, \ldots, c_{2^{k-1}0}, \ldots, c_{2^{k-1}M-1}]^T = [c_1, c_2, \ldots, c_{2^{k-1}M}]^T \qquad \ldots(7)$$

and

$$\varphi(t) = [\varphi_{10}(t), \varphi_{11}(t), \ldots, \varphi_{1M-1}(t), \ldots, \varphi_{20}(t), \varphi_{21}(t), \ldots, \varphi_{2M-1}(t), \ldots, \varphi_{2^{k-1}0}(t), \varphi_{2^{k-1}1}(t), \ldots, \varphi_{2^{k-1}M-1}(t)]^T$$
$$= [\varphi_1(t), \varphi_2(t), \ldots, \varphi_{2^{k-1}M-1}(t)]$$

...(8)

## 3. Operational Matrices of Integration [10,11, 16]

The integration of the vector $\varphi(t)$ defined in equ.(8) can be obtained as.

$$\int_0^t [\varphi(\theta)] d\theta = p\varphi(t)$$

...(9)

where p is the $2^{k-1}M \times 2^{k-1}M$ operational matrix for integration. This matrix was obtained as.

$$p = \frac{1}{2^k} \begin{bmatrix} L & F & F & \cdots & F & F \\ 0 & L & F & \cdots & F & F \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & L & F \\ 0 & 0 & 0 & \cdots & 0 & L \end{bmatrix}$$

...(10)

where F and L are M×M matrices given by:-

$$F = \begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

and

$$L = \frac{1}{2^k} \begin{bmatrix} 1 & \frac{1}{\sqrt{3}} & 0 & 0 & \cdots & 0 & 0 \\ \frac{-\sqrt{3}}{\sqrt{3}} & 0 & \frac{\sqrt{3}}{3\sqrt{5}} & 0 & \cdots & 0 & 0 \\ 0 & \frac{-\sqrt{5}}{5\sqrt{3}} & 0 & \frac{\sqrt{5}}{5\sqrt{7}} & \cdots & 0 & 0 \\ 0 & 0 & \frac{-\sqrt{7}}{7\sqrt{5}} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{\sqrt{2M-3}}{(2M-3)\sqrt{2M-1}} \\ 0 & 0 & 0 & 0 & \cdots & \frac{-\sqrt{2M-1}}{(2M-1)\sqrt{2M-3}} & 0 \end{bmatrix} \quad ...(11)$$

The integration of the product of two Legendre wavelets function vectors is obtained as:-

$$I = \int_0^1 \varphi(t)\varphi^T(t) dt$$

...(12)

where $I$ is an identity matrix

## 4. Operational Matrices of Derivative [15]

In the following steps we introduce the method for deriving Legendre wavelets operational matrix of derivative.

Theorem (1):-

Let $\varphi(t)$ represents the Legendre wavelets vector defined in equ.(8) the derivative of the $\varphi(t)$ can be expressed by

$$\frac{d\varphi(t)}{dt} = D\varphi(t) \qquad ...(13)$$

where D is the $2^k \times (M+1)$ operational matrix of derivative defined thus:-

$$D = \begin{bmatrix} F & 0 & \cdots & 0 \\ 0 & F & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & F \end{bmatrix}$$

...(14)

where F is (M+1) ×(M+1) matrix and its $(r,s)^{th}$ element is thus:

$$F_{r,s} = \begin{cases} 2^k\sqrt{(2r-1)(2s-1)} & r = 2,\dots,(M+1),\ s = 1,\dots,r-1 \text{ and } (r+s) \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

...(15)

so $D\varphi_{6\times6}$ matrix will be construction using equ.(15) as follows

$$D\varphi_{6\times6} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\sqrt{15} & 0 & 0 & 0 & 0 \\ 2\sqrt{7} & 0 & 2\sqrt{35} & 0 & 0 & 0 \\ 0 & 2\sqrt{27} & 0 & 2\sqrt{63} & 0 & 0 \\ 2\sqrt{11} & 0 & 2\sqrt{55} & 0 & 2\sqrt{99} & 0 \end{bmatrix} \qquad ...(16)$$

Corollary (1):-

By theorem (1) the operational matrix for the $n^{th}$ derivative can be derived as

$$\frac{d^n\varphi(t)}{dt^n} = D^n\varphi(t)$$ where $D^n$ is the $n^{th}$ power of the matrix D.

$$\frac{d^2\varphi(t)}{dt^2} = D^2\varphi_{6\times6}(t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 12\sqrt{5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 20\sqrt{21} & 0 & 0 & 0 & 0 \\ 40\sqrt{9} & 0 & 28\sqrt{45} & 0 & 0 & 0 \\ 36\sqrt{77} & 56\sqrt{33} & 0 & 0 & 0 & 0 \end{bmatrix} \qquad ...(17)$$

## 5. Power in Terms of Legendre Wavelets as Basis Functions

We will derive the powers in terms of Legendre wavelets which help us to solve our problems. Let take M= 6 and m= 0,1,..., 5. First, find six basis function which given by

$$\varphi_{10} = 1$$
$$\varphi_{11} = \sqrt{3(2t-1)}$$
$$\varphi_{12} = \sqrt{5(6t^2-6t+1)}$$
$$\varphi_{13} = \sqrt{7(20t^3-30t^2+12t-1)}$$
$$\varphi\_14 = \sqrt{9}(70t\text{^}4 - 140t\text{^}3 [+90t]\text{^}2 - 20t + 1$$
$$\varphi_{15} = \sqrt{11(252t^5-630t^4+560t^3-210t^2+30t-1)}$$

Hence

$$t^0 = \varphi_{10}$$
$$t^1 = \frac{\varphi_{10}}{2} + \frac{\varphi_{11}}{2\sqrt{3}}$$
$$t^2 = \frac{\varphi_{10}}{3} + \frac{\varphi_{11}}{2\sqrt{3}} + \frac{\varphi_{12}}{6\sqrt{5}}$$
$$t^3 = \frac{\varphi_{10}}{4} + \frac{9\varphi_{11}}{20\sqrt{3}} + \frac{\varphi_{12}}{4\sqrt{5}} + \frac{\varphi_{13}}{20\sqrt{7}}$$
$$t^4 = \frac{\varphi_{10}}{5} + \frac{2\varphi_{11}}{5\sqrt{3}} + \frac{2\varphi_{12}}{7\sqrt{5}} + \frac{\varphi_{13}}{10\sqrt{7}} + \frac{\varphi_{14}}{70\sqrt{9}}$$
$$t^5 = \frac{\varphi_{10}}{6} + \frac{5\varphi_{11}}{14\sqrt{3}} + \frac{25\varphi_{12}}{84\sqrt{5}} + \frac{5\varphi_{13}}{36\sqrt{7}} + \frac{\varphi_{14}}{84\sqrt{9}} + \frac{\varphi_{15}}{252\sqrt{11}}$$

In matrix form, the powers of t can be rewritten as follows

$$A = T\,B$$

where

$$T_{6\times 6} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2\sqrt{3}} & \frac{1}{6\sqrt{5}} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{9}{20\sqrt{3}} & \frac{1}{4\sqrt{5}} & \frac{1}{20\sqrt{7}} & 0 & 0 \\ \frac{1}{5} & \frac{2}{5\sqrt{3}} & \frac{2}{7\sqrt{5}} & \frac{1}{10\sqrt{7}} & \frac{1}{70\sqrt{9}} & 0 \\ \frac{1}{6} & \frac{4}{14\sqrt{3}} & \frac{25}{84\sqrt{5}} & \frac{5}{36\sqrt{7}} & \frac{1}{84\sqrt{9}} & \frac{1}{252\sqrt{11}} \end{bmatrix} \qquad ...(18)$$

where

$$A = \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \\ t^4 \\ t^5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \varphi_{10} \\ \varphi_{11} \\ \varphi_{12} \\ \varphi_{13} \\ \varphi_{14} \\ \varphi_{15} \end{bmatrix}$$

## 6. Solution of Boundary Value Problem Using Legendre Wavelets Method

Here, two cases of this problem will be studied

Case (1): In this case we will show the method of solving the boundary value problem as a differential equation.

Consider the following n$^{th}$ order boundary value problem

$$y^n(x) = f\left(x, y, \frac{dy}{dx}, \dots, \frac{d^{n-1}y}{dx^{n-1}}\right)$$

…(19)

with boundary conditions

$$y^{(k)}(a) = \gamma_k, \quad y^{(k)}(b) = \beta_k \quad \dots(20)$$

$x \in [a, b]$ and k=1,2,…,n-1

Where $f\left(x, y, \frac{dy}{dx}, \dots, \frac{d^{n-1}y}{dx^{n-1}}\right)$ and y(x) are assumed real and as many as times differentiable as required for $x \in [a, b]$.

To solving this equation by Legendre wavelets method, unknown functions y(x) are considered as a linear combination of these wavelets as follows.

$$y(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \varphi_{n,m}(x) \quad \dots(21)$$

By substituting this approximation into eq(19) we get

$$\left(\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \varphi_{n,m}(x)\right)^{(n)} = f\left(x, \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \varphi_{n,m}(x), \frac{d}{dx}\left(\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \varphi_{n,m}(x)\right), \dots, \frac{d^{n-1}}{dx^{n-1}}\left(\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \varphi_{n,m}(x)\right)\right)$$

Rearranging the above equation into the following formula

$$\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \left(\frac{d^n}{dx^n} \varphi_{n,m}(x) - f\left(\varphi_{n,m}(x), \frac{d}{dx} \varphi_{n,m}(x), \dots, \frac{d^n}{dx^n} \varphi_{n,m}(x)\right)\right) = f(x) \quad \dots(22)$$

Using the operational matrix of derivative $D^n\varphi_{n,m}(x)$ and rewriting f(x) in terms of $\varphi_{n,m}(x)$ we have a linear system of algebraic equation of $c_{n,m}$ .

By solving this system the vector function $c_{n,m}$ can be obtained

The following algorithm shows the basic steps to find the approximate solution of BVB using the first case.

**Algorithm (1)**

Inputs

n = ($2^{k-1}$ The fist index of $\varphi$ and k=1,2,...)

m = ( The order of Legendre function and it's the second index of $\varphi$ )

M = (The value which determined the parameter m since m=M-1)

Output

$c_i^{'}s$ (i= 0,1, ..., m) ( The unknown coefficients of equ.(6).

$y_{nm}(t)$ ( The approximate solution of the boundary value problem)

Step 1: Write down the Legendre wavelets as a basis function using equ.(2), $\varphi_{n,m}(x)$ for n =1 and m =0,1,2,...,M-1

Step 2: Set $y_{n,m}(x) = c_0\varphi_{10}(x) + c_1\varphi_{11}(x) + \cdots + c_m\varphi_{1m}(x)$

Step 3: Substitute $y_{n,m}(x)$ into equ.(19)

Step 4: Determine the operational matrix of derivative $D^n\varphi(x)$

Step 5: Rewriting the function f(x) in terms of $\varphi_{n,m}(x)$

Step 6: Rearrange the equ.(19) for the coefficients $c_i^{'}s$ i=0,1,...,m

Step 7: Use the solve formula to determine the value of $c_i^{'}s$

Step 8: Substitute the results of (step 7) into (step 2) so we find the approximate solution $y_{n,m}(x)$ for equ.(19)

Case (2): In this case the introduced method will be applied to solve BVP after reducing it to a system of Volterra integral equation as follows

$$y_1 = \alpha_0 + \int_0^x y_2(t)dt$$

$$y_2 = \alpha_1 + \int_0^x y_3(t)dt$$

$$y_3 = \alpha_2 + \int_0^x y_4(t)dt$$

$$\vdots$$

$$y_n = \alpha_{n-1} + \int_0^x f(t, y_1(t), y_2(t), \ldots, y_{n-1}(t))dt \qquad \ldots(23)$$

where $y = y_1, \frac{dy}{dx} = y_2, \frac{d^2y}{dx^2} = y_3, \ldots, \frac{d^{n-1}y}{dx^{n-1}} = y_n$

First we assume the unknowns functions $y_i$ for i=1,2,...,n re approximated in the following forms

$$y_i = \sum_{m=0}^{M-1} c_{1,m}^{i} \varphi_{1,m}(x) \qquad ...(24)$$

Therefore we have

$$\sum_{m=0}^{M-1} c_{1,m}^{1} \varphi_{1,m}(x) = \alpha_0 + \int_0^x \sum_{m=0}^{M-1} c_{1,m}^{2} \varphi_{1,m}(t)dt$$

$$\sum_{m=0}^{M-1} c_{1,m}^{2} \varphi_{1,m}(x) = \alpha_1 + \int_0^x \sum_{m=0}^{M-1} c_{1,m}^{3} \varphi_{1,m}(t)dt$$

$$\vdots$$

$$\sum_{m=0}^{M-1} c_{1,m}^{n} \varphi_{1,m}(x) = \alpha_{n-1} + \int_0^x f\left(t, \sum_{m=0}^{M-1} c_{1,m}^{1} \varphi_{1,m}(t), \sum_{m=0}^{M-1} c_{1,m}^{2} \varphi_{1,m}(t), ..., \sum_{m=0}^{M-1} c_{1,m}^{n-1} \varphi_{1,m}(t)\right)dt \qquad ...(25)$$

Using t power ($\alpha_i$) and f(t) will be considered as a linear combination of Legendre wavelets. To integrate $\varphi_{i,m}(t)$ in the right side the operational matrix of integration equ.(10) will be used, the result will be a system of algebraic equation for both $\varphi_{1,m}(x)$ *and* $c_{1,m}^{i}$ and by equivalent the coefficients of $\varphi_{1,m}(x)$ a system of linear equations in terms of the entries $c_{1,m}^{i}$ will be obtained and the elements of vectors $c_{1,m}^{i}$ will be found by solving this system.

The following algorithm shows the simple steps to find the vector $c_{1,m}^{i}$ for the approximate solution in case (2).

**Algorithm (2)**

Step 1:- Input ( $\alpha_i$ ) for (i= 0,1,...,n-1)
Step 2:- Input (n,M)
Step 3:- Select the Legendre wavelets as a basis functions $\varphi_{n,m}(t)$
Step 4:- Put $y_n = c_{1,0}^{1}\varphi_{10}(x) + ... + c_{1,m}^{n}\varphi_{1,m}(x)$
Step 5:- Put $f(t) = \varphi_{10}(x) + \varphi_{11}(x) + ... + \varphi_{1,m}(x)$
Step 6:- Integrate $y_n$ in step 4 using operational matrix of integration
Step 7:- If ( $i \neq n$) then D = $y_i$ – the result in step 6 for (i+1)
Step 8:- If (i = n) then D = $y_i$ – the result in step 6 for (i=n)
Step 9:- Let $\beta_{ij}$ = the coefficients of $\varphi_{n,m}(t)$ in step (7 and 8)
Step 10:- Solve the equation $\beta_{ij} = \alpha_{i-1}$ and $\beta_{nj}$ = f(t)+ $\alpha_{n-1}$ for $c_{1,m}^{i}$
Step 11:- Substitute the results of step 10 into step 4 to find the approximate solution of the problem

## 7. Numerical Examples

In this section, to demonstrate the effectiveness of the proposed algorithms, two examples are discussed.

Example 1:-

Consider the following boundary value problem $y^{(3)}(t)-y(t)=5-t^3$ with the boundary condition $y(0)=1, y'(0)=0, y''(0)=0$ and $y(1)=2, y'(1)=3, y''(1)=6$

with the exact solution $y(t)=t^3+1$

For solving this problem using first approach we apply the following steps

Assume $y(t)\cong y_3(t)=\sum_{m=0}^{3} c_m\varphi_{1,m}(t)$

So, $y(t)\cong y_3(t)=c_0+c_1\sqrt{3}(2t-1)+c_2\sqrt{5}(6t^2-6t+1)+c_3\sqrt{7}(20t^3-30t^2+12t-1)$

Substitute the above equation into the problem and using the operational matrix of derivative we get :-

$$y^{(3)}(t)=120\sqrt{7}\varphi_{10}(t)c_3$$

Also for $f(t)=5-t^3$ we use (t power ) the transforming will be

$$5-t^3=5\varphi_{10}-\left(\frac{\varphi_{10}}{4}+\frac{9\varphi_{11}}{20\sqrt{3}}+\frac{\varphi_{12}}{4\sqrt{5}}+\frac{\varphi_{13}}{20\sqrt{7}}\right)$$

Hence

$$120\sqrt{7}\varphi_{10}(t)c_3=5\varphi_{10}-\left(\frac{\varphi_{10}}{4}+\frac{9\varphi_{11}}{20\sqrt{3}}+\frac{\varphi_{12}}{4\sqrt{5}}+\frac{\varphi_{13}}{20\sqrt{7}}\right)+c_0\varphi_{10}+c_1\varphi_{11}+c_2\varphi_{12}+c_3\varphi_{13}$$

From the equivalent of the term $\varphi_{1,m}(t)$ in this equation we obtain four linear equations for $c_m$'s as follows

$$y(t)=1+t^3$$

$$c_1-\frac{9}{20\sqrt{3}}=0 \quad then \quad c_1=\frac{9}{20\sqrt{3}}$$

$$c_2-\frac{1}{4\sqrt{5}}=0 \quad then \quad c_2=\frac{1}{4\sqrt{5}}$$

$$c_3-\frac{1}{20\sqrt{7}}=0 \quad then \quad c_3=\frac{1}{20\sqrt{7}}$$

The approximate solution is $y(t)\cong y_3(t)=1+t^3$

To solve the same example using second approach we following the steps

Transforming the differential equation into a system of four integral equations as follows:-

$$y=y_1,\ \frac{dy}{dt}=y_2,\ \frac{d^2y}{dt^2}=y_3$$

So

$$y_1 = 1 + \int_0^t y_2(x)dx$$

$$y_2 = \int_0^t y_3(x)dx$$

$$y_3 = \int_0^t (5 - x^3)dx + \int_0^t y_1(x)dx$$

Assume that $y_i = \sum_{m=0}^{3} c_{1,m}^i \varphi_{1,m}(t)$ for (i=1, 2, 3)

Substitute this function into the system the result is:-

$$\sum_{m=0}^{3} c_{1,m}^1 \varphi_{1,m}(t) = \varphi_{10} + \int_0^t \sum_{m=0}^{3} c_{1,m}^2 \varphi_{1,m}(x)dx$$

$$\sum_{m=0}^{3} c_{1,m}^2 \varphi_{1,m}(t) = \int_0^t \sum_{m=0}^{3} c_{1,m}^3 \varphi_{1,m}(x)dx$$

$$\sum_{m=0}^{3} c_{1,m}^3 \varphi_{1,m}(t) = \int_0^t (5 - x^3)dx + \int_0^t \sum_{m=0}^{3} c_{1,m}^1 \varphi_{1,m}(x)dx$$

By using (t power) the last equation will be written as:-

$$\sum_{m=0}^{3} c_{1,m}^3 \varphi_{1,m}(t) = 5\left(\frac{\varphi_{10}}{2} + \frac{\varphi_{11}}{2\sqrt{3}}\right) - \frac{1}{4}\left(\frac{\varphi_{10}}{5} + \frac{2\varphi_{11}}{5\sqrt{3}} + \frac{\varphi_{13}}{10\sqrt{7}} + \frac{\varphi_{14}}{70\sqrt{9}}\right) + \int_0^t \sum_{m=0}^{3} c_{1,m}^1 \varphi_{1,m}(x)dx \quad ...(26)$$

By solving the integration in the right side of each equation using operational matrix of integration and gathering the terms which contain the coefficients $c_{1,m}^i$ and equivalent with functions $\varphi_{1,m}$ in the other side we get the following system of linear equations

$$c_0^1 - \frac{c_0^2}{2} + \frac{\sqrt{3}}{6}c_1^2 = 1 \quad , \quad c_1^1 - \frac{c_0^2}{2\sqrt{3}} + \frac{\sqrt{5}}{10\sqrt{3}}c_2^2 = 0$$

$$c_2^1 - \frac{\sqrt{3}c_1^2}{6\sqrt{5}} + \frac{\sqrt{7}}{14\sqrt{5}}c_3^2 = 0 \quad , \quad c_3^1 - \frac{\sqrt{5}c_2^2}{10\sqrt{7}} = 0 \quad , \quad \frac{\sqrt{7}c_3^2}{14\sqrt{9}} = 0$$

$$c_0^2 - \frac{c_0^3}{2} + \frac{\sqrt{3}}{6}c_1^3 = 0 \quad , \quad c_1^2 - \frac{c_0^3}{2\sqrt{3}} + \frac{\sqrt{5}}{10\sqrt{3}}c_2^3 = 0$$

$$c_2^2 - \frac{\sqrt{3}c_1^3}{6\sqrt{5}} + \frac{\sqrt{7}}{14\sqrt{5}}c_3^3 = 0 \quad , \quad c_3^2 - \frac{\sqrt{5}c_2^3}{10\sqrt{7}} = 0 \quad , \quad \frac{\sqrt{7}c_3^3}{14\sqrt{9}} = 0$$

$$c_0^3 - \frac{c_0^1}{2} + \frac{\sqrt{3}}{6}c_1^1 = \frac{49}{20} \quad , \quad c_1^3 - \frac{c_0^1}{2\sqrt{3}} + \frac{\sqrt{5}}{10\sqrt{3}}c_2^1 = \frac{24}{10\sqrt{3}}$$

$$c_2^3 - \frac{\sqrt{3}c_1^1}{6\sqrt{5}} + \frac{\sqrt{7}}{14\sqrt{5}}c_3^1 = \frac{-1}{14\sqrt{5}} \;, \quad c_3^3 - \frac{\sqrt{5}c_2^1}{10\sqrt{7}} = \frac{-1}{40\sqrt{7}} \;, \quad \frac{\sqrt{7}\,c_3^1}{14\sqrt{9}} = \frac{1}{280\sqrt{9}}$$

By using mathlab program to solve this system we get the results in table (1)

Table (1)

| m | $C_m^1$ | $C_m^2$ | $C_m^3$ |
|---|---|---|---|
| 0 | 1/4 | 1 | 3 |
| 1 | 9/20√3 | 3/2√3 | 3/√3 |
| 2 | 1/4√5 | 1/2√5 | 0 |
| 3 | 1/20√7 | 0 | 0 |
| Y(t) | $Y_1(t)=1+t^3$ | $Y_2(t)=3t^2$ | $Y_3(t)=6t$ |

Approximate values of $y_m(t)$ and $C_m^i$

## Example (2):-

Consider the boundary value problem $y^{(4)}(t)-y(t)+t^4-24=0$
with the conditions $y(0)=0$ *and* $y^{(j)}(0)=0$ for (j=1, 2, 3)

$y(1)=1$ , $y'(1)=4, y''(1)=12, y'''(1)=24$

The exact solution is $y(t)=t^4$

Assume that $y(t)\cong y_4(t)=\sum_{m=0}^{4} c_m \varphi_{1,m}(t)$

So , $y(t)\cong y_4(t)=c_0\varphi_{10}+c_1\varphi_{11}+c_2\varphi_{12}+c_3\varphi_{13}+c_4\varphi_{14}$

By substituting the value of $y_4(t)$ in the boundary equation and following the steps in the first approach such as in the first example the approximate coefficients value will be:-

$$c_0=\frac{1}{5},c_1=\frac{2}{5\sqrt{3}},c_2=\frac{2}{7\sqrt{5}},c_3=\frac{1}{10\sqrt{7}},c_4=\frac{1}{210}$$

So the approximate solution is

$$y(t)\cong y_4(t)=\frac{1}{5}+\frac{2}{5\sqrt{3}}\sqrt{3}(2t-1)+\frac{2}{7\sqrt{5}}\sqrt{5}(6t^2-6t+1)+\frac{1}{10\sqrt{7}}\sqrt{7}(20t^3-30t^2+12t-1)+\frac{1}{210}\sqrt{9}(70t^4-140t^3+90t^2-20t+1)$$ y

ield $y(t)\cong y_4(t)=t^4$

For solving this example using the second approach the same steps fulfilled in the first example will be carryout in this example to solve a system of four equations the result is showed in table (2).

Table(2)

| m | $C_m^1$ | $C_m^2$ | $C_m^3$ | $C_m^4$ |
|---|---|---|---|---|
| 0 | 1/5 | 1 | 4 | 12 |
| 1 | 2/5√3 | 9/5√3 | 6/√3 | 12/√3 |

| | | | | |
|---|---|---|---|---|
| 2 | $2/7\sqrt{5}$ | $1/\sqrt{5}$ | $2/\sqrt{5}$ | 0 |
| 3 | $1/10\sqrt{7}$ | $1/5\sqrt{7}$ | 0 | 0 |
| 4 | 1/210 | 0 | 0 | 0 |
| Y(t) | $y_1(t)=t^4$ | $y_2(t)=4t^3$ | $y_3(t)=12t^2$ | $y_4(t)=24t$ |

Approximate values of $y_m(t)$ and $C_m^i$

## 8. Conclusion

Legendre wavelets method has been used to derive approximate solutions for $n^{th}$ order boundary value problems. The obtained results show that the method is very powerful and useful technique for finding the approximate solution of such problems. Comparison of the approximate solutions and the exact solutions show that the proposed method is more practical for solving $n^{th}$ order BVP. Directly, that is when the proposed algorithm (1) was used.

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## طريقة لجندر الموجية لحل مسائل القيم الحدودية

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المستخلص

تم تقديم اسلوبين لحل مسائل القيمة الحدودية من الرتبة النونية باستخدام متعددات لجندر الموجية المستمرة ضمن الفترة [0, 1]. الخوارزمية الاولى تحل المسألة مباشرة باستخدام مصفوفة العمليات المشتقة لمتعددات لجندر الموجية بينما الخوارزمية الثانية تحول المسألة الى منظومة من معادلات فولتيرا التكاملية ومن ثم استخدام مصفوفة العمليات التكاملية لنفس المتعددات، منظومة المعادلات التكاملية تختزل الى حل مجموعة من المعادلات الجبرية الخطية ، بعض الامثلة قدمت لبيان كفاءة الخوارزميات .