Some results on Fuzzy Pre- Hilbert Space

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Abstract:

The aim of this work is to introduce new definition of fuzzy pre-Hilbert space (fuzzy inner product space) and discussed the relation between fuzzy pre-Hilbert space and ordinary pre-Hilbert space.

Much attention is paid to the concepts of convergence of sequence (for example Cauchy sequence of fuzzy point) and the concept of boundness of these sequences.

Introduction:

In 1965 Zadeh [8] mathematically formulated the fuzzy subset concept. He defined fuzzy subset of a non-empty set as a collection of objects with grade of membership in a continuum, with each object being assigned a value between 0 and 1 by a membership function.

J.R. Kider was introduced new definition of fuzzy normed space [4], which are using in section two of this paper.

The aim of section one is to cover the basic concepts of fuzzy sets.

In section two we introduced the definition of fuzzy pre-Hilbert space and discusse properties of this space ,also we prove every fuzzy pre- Hilbert space **(P,**$(.|.)_f$) is afuzzy normed space by defining

$$||x_\propto||_f = (x_\propto|x_\propto)^{\frac{1}{2}}.$$

The convergences of sequences of fuzzy point are discusse in section three.

S1:Basic concepts aboute fuzzy sets

Definition 1.1: [8],[9]

Let X be any set of element.A fuzzy set $\tilde{A}\, in\, X$ is characterized by a membership function, $\mu_{\tilde{A}}(x){:}X \to I, where\, I$ is the closed unit interval [0,1].Then we can write a fuzzy set $\tilde{A}$ as:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x))|x \in X, 0 \le \mu_{\tilde{A}}(x) \le 1\}.$$

i.e afuzzy set $\tilde{A}$ is written as aset of pairs (x, $\mu_{\tilde{A}}(x)$).

Where x is an element of universal set X and the value $\mu_{\tilde{A}}(x)$ is the degree of membership of element x in afuzzy set $\tilde{A}$.

Definition 1.2[2]:

Afuzzy set $\tilde{A}$ is called normal if there is at least one point $x \in X$ with $\mu_{\tilde{A}}(x)$=1.

Remark 1.3: [1]

Each ordinary set A is a fuzzy set with a membership function defined by:

$$\mu_{\tilde{A}}(x) = \begin{cases} 1, & x \in A \\ 0, & otherwise \end{cases}$$

For each $x \in X$.

Remarks 1.4: [1], [6]

Following, some fundamental concept related to the basic operations and concepts of fuzzy subsets of X.

Let $\tilde{A}\ and\ \tilde{B}$ be two fuzzy subsets of X with membership function $\mu_{\tilde{A}}\ and\ \mu_{\tilde{B}}$ respectively

1- $\tilde{A} \subseteq \tilde{B}$ if and only if $\mu_{\tilde{A}}(x) \le \mu_{\tilde{B}}(x), \quad \forall x \in X$

2- $\tilde{A} = \tilde{B}\ if\ and\ only\ if\ \mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x),\ \forall x \in X.$

3- The complement of $\tilde{A}$ (denoted by $\tilde{A}^C$) is also a fuzzy set which has the membership function $\mu_{\tilde{A}^C}(x) = 1 - \mu_{\tilde{A}}(x)$.

4-$\tilde{A} = \emptyset\ if\ and\ only\ if\ \mu_{\tilde{A}}(x) = 0\ , \forall x \in X\ , where\ \emptyset\ is\ the\ empty\ fuzzy\ set.$

5- The height of fuzzy set is the supremum value of $\mu_{\tilde{A}}(x) over\ all\ x \in X$. If the height is 1, then $\tilde{A}$ is normal, otherwise it is subnormal.

6- The support of $\tilde{A}$ is the set of all elements $x\ in\ X\ at\ which$ $0 < \mu_{\tilde{A}}(x) \le 1$ and is denoted by supp $(\tilde{A})$.

7- A point $x \in X$ is said to be crossover point of $\tilde{A}\ if\ \mu_{\tilde{A}}(x) = 0.5$.

8- $\tilde{C} = \tilde{A} \cap \tilde{B}$ is a fuzzy set with membership function

$\mu_{\tilde{C}}(x) = \min_{x\in X}\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$.

9- $\tilde{D} = \tilde{A} \cup \tilde{B}$ is a fuzzy set with membership function

$\mu_{\tilde{D}}(x) = \max_{x\in X}\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$.

10- If $\mu_{\tilde{A}\cap\tilde{B}}(x) = 0, \forall x \in X, then\ \tilde{A}\ and\ \tilde{B}$ are said to be separated fuzzy sets.

11- The product of $\tilde{A}\ and \tilde{B}$ (denoted by $\tilde{A}\tilde{B}$) is also fuzzy set which have a membership function $\mu_{\tilde{A}\tilde{B}}(x) = \mu_{\tilde{A}}(x)\mu_{\tilde{B}}(x)$.

12- The n-th power of $\tilde{A}$(denoted by $\tilde{A}^n$) is also a fuzzy set which have a membership function $\mu_{\tilde{A}^n} = [\mu_{\tilde{A}}(x)]^n$ where n is a positive integer and consequently

$\tilde{A}^n \subseteq \tilde{A}^m$ for all $n \geq m \geq 0$.

Definition 1.5: [7]

A fuzzy point $p\ in\ X$ is a fuzzy set with membership function.

$$P(y) = \begin{cases} \alpha, & if\ y = x \\ 0, & otherwise \end{cases}$$

For all $y\ in\ X$ where $0 < \alpha < 1. P$ is said to have support x and value α. (x is fixed point).We denote this fuzzy point by x_α $or\ (x, \alpha)$.

$Two\ fuzzy\ points\ x_\alpha and\ y_\delta$ are said to be distinct if and only if $x \neq y$.

Remark 1.6[3]:

Two fuzzy points $x_\alpha\ and\ y_\beta$ are said to be equal if and only if $x = y\ and\ \alpha = \beta, where\ \alpha, \beta \in (0,1]$.

Definition 1.7[4]:

Let X be a vector space over field K (k = R or K = C) . Let $X \to [0, \infty) be\ a\ function\ which\ assigns\ to\ each\ point\ x_\alpha\ in\ X, \alpha \in (0,1]\ anonnegative\ real\ number\ ||x_\alpha||_f\ such\ that$

(FN1) $||x_\alpha||_f = 0\ if\ and\ only\ if\ x = 0$

(FN2) $||\lambda x_\alpha||_f = |\lambda|\,||x_\alpha||_f\ for\ all\ \lambda \in K$

(FN3) $\left\|x_\alpha + y_\beta\right\|_f \le \left\|x_\alpha\right\|_f + \left\|y_\beta\right\|_f$

(FN4) if $\left\|x_\alpha\right\|_f < r \; where \; r > 0 \; then \; there \; exists \; 0 < \sigma \le \alpha < 1 \; such \; that \; \left\|x_\alpha\right\|_f < r.$

Then $\left\|.\right\|_f \; is \; called \; fuzzy \; norm \; and \; \left(X, \left\|.\right\|_f\right) is \; called \; fuzzy \; normed \; space.$

S2: Fuzzy pre-Hilbert space

Definition 2.1:[5]

Apre- Hilbert space is acomplex vector space P. For each pair of Vectors x , y of P there is determined acomplex number called the Scalar product of x and y,denoted (x|y).scalar products are assumed to obey these rules:

(p1) (y|x)=$(x|y)^*$ denoted the conjugate of complex number)(*

(p2) (x+y|z)=(x|z)+(y|z)

(p3) (λx|y)= λ(x|y)

(p4) (x|x) >0 when x$\neq$ 0

Definition 2.2:

Let P be avector space over field F (F =R or F=C) , let (.|.)_f : $P \to [0,\infty)$ be afunction which assigns to every pair vectors x_∞ and y_β in P , $\infty, \beta \in (0,1]$,there is associated ascalar (x_∞/y_β)_f such that:

(FIP1) (x_∞ + z_σ/y_β)_f = (x_∞/y_β)_(f) + (z_σ/y_β)_f **where** $z_\sigma \in P$,$\sigma \in (0,]$

(FIP2) (cx_∞/y_β)_f = c(x_∞/ y_β)_f **where** c$\in$ F

(FIP3) (x_∞/y_β)_f = (y_β|x_∞)^(∗) (∗ **denoted the conjugate of complex number)**

(FIP4) (x_∞/x_∞)_f $\ge$ 0 and (x_∞/x_∞)_f = 0 $\leftrightarrow$ x = 0

(FIP5) If (x_∞/x_∞)_f < r **where r**$>$ 0 **then there exists** $0 < \infty \le \beta < 1$ **such that** (x_β|x_β)_f < r **then (.|.)_f is called fuzzy Pre-Hilbert and (P,(.|.)_f) is called fuzzy Pre-Hilbert space.**

Proposition 2.3

Let $(P,(.|.))$ **be an ordinary Pre-Hilbert space. Define** $(x_\alpha|y_\beta)_f = \frac{1}{\lambda}(x|y)$ **for every** $x_\alpha, y_\beta \in P$ **,where** $\alpha, \beta \in (0,1]$ **and** $\lambda = \min\{\alpha, \beta\}$ **then** $(P,(.|.)_f)$ **is a fuzzy Pre –Hilbert Space.**

Proof:

Let $x_\alpha, y_\beta, z_\sigma \in P$ where $\alpha, \beta, \sigma \in (0,1]$ and $c \in F$ then :

(FIP1) (x_α + z_σ ⫞| y_β)_f = (x + z|y)_f (λ)

$$= \frac{1}{\lambda}(x+z|y)$$

$$= \frac{1}{\lambda}[(x|y) + (z|y)]$$

$= \frac{1}{\lambda}(x|y) + \frac{1}{\lambda}(z|y)$

$= (x_\alpha|y_\beta)_f + (z_\sigma|y_\beta)_f$

(FIP2) $(cx_\alpha/y_\beta) = (cx|y)_f(\lambda)$

$= \frac{1}{\lambda}(cx|y)$

$= \frac{1}{\lambda}\,[c\,(x|y)\,]$

$= c[\,\frac{1}{\lambda}\,(x|y)\,]$

$= c(x_\alpha/y_\beta)_f$

(FIP3) $(x_\alpha/y_\beta)_f = (x/y)_f(\lambda)$

$= 1/\lambda(x|\ y)$

$= \frac{1}{\lambda}(y/x)^*$

$= (y_\beta/x_\alpha)_f^*$

(FIP4) Since($x|x) \geq 0$ so $\frac{1}{\alpha}(x|x) \geq 0$ thus $(x_\alpha|x_\alpha)_f \geq 0$ also $(x_\alpha|x_\alpha)_f = 0$ $= \frac{1}{\alpha}(x|x) \leftrightarrow (x|x) = 0 \leftrightarrow x = 0$

(FIP5) If $(x_\alpha|x_\alpha)_f < r$ where $r > 0$ then for $\beta \in (0,1]$ with $\alpha \leq \beta$ where

$\frac{(x|x)}{\beta} \leq \frac{(x|x)}{\alpha} < r$ that is $(x_\beta|x_\beta)_f < r$

Example 2.4:

Let $P = L^2$ [the space of all complex (real) sequence]
is an Pre-Hilbert space with Pre-Hilbert defined by L^2

$(x|y) = \sum_{j=1}^{\infty} x_j\bar{y}_j \quad \forall\ x = \{x_j\}_{j=}^{\infty} \quad y = \{y_j\}_{j=1}^{\infty}$ hence $(L^2, (.|.)_f)$ is fuzzy Pre-Hilbert Space with fuzzy Pre –Hilbert defined by $(x_\alpha|y_\beta)_f = \frac{1}{\lambda}(x|y) \quad \lambda = \min\{\alpha,\beta\} \quad \alpha,\beta \in (0,1] \quad \forall x_\alpha, y_\beta \in L^2$

The prove of the following proposition is clear ,hence is omitted.

Proposition2.5:

If $(P,(.|.)_f)$ is a fuzzy Pre-Hilbert Space then (P,(.|.)) is an ordinary Pre-Hilbert Space with ,$(x|y) = ((x,1)|(y,1))_f$,$\forall\ x, y \in P$

Theorem2.6:

In afuzzy Pre-Hilbert Space $(P,(.|.)_f)$

(a) $(x_\alpha|y_\beta + z_\sigma)_f = (x_\alpha|y_\beta)_f + (x_\alpha|z_\sigma)_f \quad \forall x_\alpha, y_\beta, z_\sigma \in P$

(b) $(x_\alpha|cy_\beta)_f = c^*(x_\alpha|y_\beta)_f \quad \forall\ c \in F$

(c) $(\theta|y_\beta)_f = (x_\alpha|\theta)_f = 0$

(d) $(x_\alpha - y_\beta | z_\sigma)_f = (x_\alpha | z_\sigma)_f - (y_\beta | z_\sigma)_f$

(e) if $(x_\alpha | z_\sigma)_f = (y_\beta | z_\sigma)_f$ for all z_σ necessarily $x_\alpha = y_\beta$

Proof **(a)**

By using (FIP1) and (FIP3)

$(x_\alpha | y_\beta + z_\sigma)_f = (y_\beta + z_\sigma | x_\alpha)_f^*$

$= [(y_\beta | x_\alpha)_f + (z_\sigma | x_\alpha)_f]^*$

$= (y_\beta | x_\alpha)_f^* + (z_\sigma | x_\alpha)_f^*$

$= (x_\alpha | y_\beta)_f + (x_\alpha | z_\sigma)_f$

(b)- Using (FIP2) and (FIP3)

$(x_\alpha | c y_\beta)_f = (c y_\beta | x_\alpha)_f^*$

$= [c (y_\beta | x_\alpha)_f]^*$

$= c^* (y_\beta | x_\alpha)_f^*$

$= c^* (x_\alpha | y_\beta)_f$

(c) $(\theta | y_\beta)_f = (0 + 0 | y_\beta)_f = (0 | y_\beta)_f + (0 | y_\beta)_f$ hence $(0 | y_\beta)_f = 0$

Similarly $(x_\alpha | \theta)_f = 0$

(d)- $(x_\alpha - y_\beta | z_\sigma)_f = (x_\alpha + (-y_\beta) | z_\sigma)_f$

$= (x_\alpha | z_\sigma)_f + (-y_\beta | z_\sigma)_f$

$= (x_\alpha | z_\sigma)_f + ((-1) y_\beta | z_\sigma)_f$

$= (x_\alpha | z_\sigma)_f - (y_\beta | z_\sigma)_f$

(e)-Suppose $(x_\alpha | z_\sigma)_f = (y_\beta | z_\sigma)_f \ \forall z_\sigma \in P$ then

$(x_\alpha - y_\beta | z_\sigma)_f = (x_\alpha | z_\sigma)_f - (y_\beta | z_\sigma)_f = 0$, in particular $(x_\alpha - y_\beta | x_\alpha - y_\beta)_f = 0$ hence $x_\alpha - y_\beta = 0$ by (FIP4)

Definition2.7:

In afuzzy Pre-Hilbert Space $(P, (\cdot|\cdot)_f)$ the fuzzy norm of vector x_α in P define by

$$\|x_\alpha\|_f = (x_\alpha | x_\alpha)^{(1/2)}$$

Theorem2.8: (Fuzzy Schwarz inequality)

In afuzzy Pre-Hilbert Space $(P, (\cdot|\cdot)_f)$

$$|(x_\alpha | y_\beta)_f| \le \|x_\alpha\|_f . \|y_\beta\|_f$$

Proof:

If $x_\alpha = 0$ or $y_\beta = 0$ then $(x_\alpha | y_\beta) = 0$ and the conclusion is clear.

Suppose, for instance ,that $y_\beta \neq 0$. Dividing through the desired inequality by $\|y_\beta\|_f$ the problem is to show that

$|(x_\alpha | z_\sigma)_f| \le \|x_\alpha\|_f$ when $|(|z_\sigma|)| = 1$

For every complex number $c \in F$

$\|x_\alpha - c z_\sigma\|_f^{2} = \|x_\alpha\|_f^2 - c^* (x_\alpha | z_\sigma)_f - c (z_\sigma | x_\alpha)_f + c c^* \|z_\sigma\|_f^2$

$= \|x_\alpha\|_f^2 - (x_\alpha|z_\sigma)_f c^* - c(x_\alpha|z_\sigma)_f^* + cc^*$

$= \|x_\alpha\|_f^2 - (x_\alpha|z_\sigma)_f (x_\alpha|z_\sigma)_f^* + (x_\alpha|z_\sigma)_f (x_\alpha|z_\sigma)_f - (x_\alpha|z_\sigma)_f c^* - c(x_\alpha|z_\sigma)_f^* + cc^*$

$= \|x_\alpha\|_f^2 - |(x_\alpha|z_\sigma)_f|^2 + [(x_\alpha|z_\sigma)_f - c][(x_\alpha|z_\sigma) - c]^*$

$= \|x_\alpha\|_f^2 - |(x_\alpha|z_\sigma)_f|^2 + |(x_\alpha|z_\sigma)_f - c|^2$

I n particular

For $c_\circ = (x_\alpha|z_\sigma)_f$

$0 \le \|x_\alpha - c_\circ z_\sigma\|_f^2 = \|x_\alpha\|_f^2 - |(x_\alpha|z_\sigma)_f|^2$

Theorem 2.9:

Every fuzzy pre-Hilbert space $(P,(\cdot|\cdot)_f)$ *is a* fuzzy normed space *by* ***defining*** $\|x_\alpha\|_f = \sqrt{(x_\alpha|x_\alpha)_f}$

Proof:

For each $x_\alpha \in P$, where $\alpha \in (0,1]$

(FN1) $\|x_\alpha\|_f = 0 = [(x_\alpha|x_\alpha)_f]^{\frac{1}{2}} \leftrightarrow (x_\alpha|x_\alpha)_f = 0 \leftrightarrow \frac{1}{\alpha}(x|x) = 0 \leftrightarrow x = 0$

(FN2) $\|cx_\alpha\|_f = [(cx_\alpha|cx_\alpha)_f]^{1/2} \ \forall c \in F$

$= [cc^*(x_\alpha|x_\alpha)_f]^{\frac{1}{2}}$

$= [|c|^2 (x_\alpha|x_\alpha)_f]^{\frac{1}{2}}$

$= |c| \, [(x_\alpha|x_\alpha)_f]^{\frac{1}{2}}$

$= |c| \, \|x_\alpha\|_f$

(FN3) $\|x_\alpha + y_\beta\|_f^2 = (x_\alpha + y_\beta|x_\alpha + y_\beta)_f \ \forall x_\alpha, y_\beta \in P$

$= (x_\alpha|x_\alpha + y_\beta)_f + (y_\beta|x_\alpha + y_\beta)_f$

$= (x_\alpha + y_\beta|x_\alpha)_f^* + (x_\alpha + y_\beta|y_\beta)_f^*$

$= (x_\alpha|x_\alpha)_f^* + (y_\beta|x_\alpha)_f^* + (x_\alpha|y_\beta)_f^* + (y_\beta|y_\beta)_f^*$

$= (x_\alpha|x_\alpha)_f + (x_\alpha|y_\beta)_f + (x_\alpha|y_\beta)_f^* + (y_\beta|y_\beta)_f$

$= \|x_\alpha\|_f^2 + 2\mathrm{Re}(x_\alpha|y_\beta)_f + \|y_\beta\|_f^2$

$\le \|x_\alpha\|_f^2 + 2|(x_\alpha|y_\beta)_f|^2 + \|y_\beta\|_f^2$

$\le \|x_\alpha\|_f^2 + 2\|x_\alpha\|_f \|y_\beta\|_f + \|y_\beta\|_f^2$

$\|x_\alpha + y_\beta\|_f^2 \le [\|x_\alpha\|_f + \|y_\beta\|_f]^2$

Hence $\|x_\alpha + y_\beta\|_f \le \|x_\alpha\|_f + \|y_\beta\|_f$

(FN4) If $\|x_\alpha\|_f < r$ where $r > 0$ then for $\sigma \in (0,1]$ with $\alpha \le \sigma$

We have $[(x_\sigma|x_\sigma)_f]^{(1/2)} \le [(x_\alpha|x_\alpha)]^{(1/2)} < r$ that is $\|x_\alpha\|_f < r$

Thus $(P,\|\cdot\|_f)$ is afuzzy normed space

In the following theorem we will show that the fuzzy norm $\|x_\alpha\|_f = [(x_\alpha|x_\alpha)_f]^{\frac{1}{2}}$ satisfies the fuzzy parallelogram equality

Theorem 2.10:

In any fuzzy pre-Hilbert space $(P,(\cdot|\cdot)_f)$

$$\|x_\alpha + y_\beta\|_f^2 + \|x_\alpha - y_\beta\|_f^2 = 2\|x_\alpha\|_f^2 + 2\|y_\beta\|_f^2 \quad \forall x_\alpha, y_\beta \in P, \ \alpha,\beta \in (0,1]$$

Proof:

One has

$$\|x_\alpha + y_\beta\|_f^2 = (x_\alpha + y_\beta | x_\alpha + y_\beta)_f$$
$$= (x_\alpha|x_\alpha)_f + (x_\alpha|y_\beta)_f + (y_\beta|x_\alpha)_f + (y_\beta|y_\beta)_f$$
$$= \|x_\alpha\|_f^2 + (x_\alpha|y_\beta)_f + (y_\beta|x_\alpha)_f + \|y_\beta\|_f^2 \quad \ldots\ldots\ldots\ldots (1)$$

Replacing (y_β) by $(-y_\beta)$

$$\|x_\alpha - y_\beta\|_f^2 = (x_\alpha - y_\beta | x_\alpha - y_\beta)_f$$
$$= (x_\alpha|x_\alpha)_f - (x_\alpha|y_\beta)_f - (y_\beta|x_\alpha)_f + (y_\beta|y_\beta)_f$$
$$= \|x_\alpha\|_f^2 - (x_\alpha|y_\beta)_f - (y_\beta|x_\alpha)_f + \|y_\beta\|_f^2 \quad \ldots\ldots\ldots\ldots (2)$$

By adding these two equation (1) and (2) we get

$$\|x_\alpha + y_\beta\|_f^2 + \|x_\alpha - y_\beta\|_f^2 = 2\|x_\alpha\|_f^2 + 2\|y_\beta\|_f^2$$

Theorem 2.11:

Let $(P,(\cdot|\cdot)_f)$ be a fuzzy be a fuzzy pre-Hilbert space, then the fuzzy **norm** $\|x_\alpha\|_f = [(x_\alpha|x_\alpha)_f]^{\frac{1}{2}}$ satisfies the fuzzy triangle inequality

$$\|x_\alpha + y_\beta\|_f \le \|x_\alpha\|_f + \|y_\beta\|_f$$

Proof:

Applying the fuzzy Schwarz inequality

$$\|x_\alpha + y_\beta\|_f^2 = \|x_\alpha\|_f^2 + \|y_\beta\|_f^2 + (x_\alpha|y_\beta)_f + (x_\alpha|y_\beta)_f^*$$
$$= \|x_\alpha\|_f^2 + \|y_\beta\|_f^2 + 2\mathrm{Re}(x_\alpha|y_\beta)_f$$
$$\le \|x_\alpha\|_f^2 + \|y_\beta\|_f^2 + 2|(x_\alpha|y_\beta)_f|$$
$$\le \|x_\alpha\|_f^2 + \|y_\beta\|_f^2 + 2\|x_\alpha\|_f\|y_\beta\|_f$$
$$= (\|x_\alpha\|_f + \|y_\beta\|_f)^2$$

Thus $\|x_\alpha + y_\beta\|_f \le \|x_\alpha\|_f + \|y_\beta\|_f$

S3: Converges ,Cauchy Fuzzy Sequence

Definition 3.1:

A fuzzy sequence $\{(x_n,\alpha_n)\}$ in afuzzy pre-Hilbert space $(P,(\cdot|\cdot)_f)$ is said to be fuzzy convergent if there is afuzzy vector x_α in P such that

$$\lim_{n\to\infty}\|(x_n,\alpha_n) - x_\alpha\|_f = 0$$

Where $\|x_\alpha\|_f = [(x_\alpha|x_\alpha)_f]^{\frac{1}{2}}$ *and* $\{\alpha, \alpha_i \in (0,1] | i \in N\}$

Definition 3.2:

A fuzzy sequence $\{(x_n, \alpha_n)\}$ in afuzzy pre-Hilbert space $(P,(\cdot|\cdot)_f)$ is said to be fuzzy Cauchy sequence if for every $\varepsilon > 0$, there is an integer $N > 0$
Such that

$\|(x_m, \alpha_m) - (x_n, \alpha_n)\|_f < \varepsilon \quad \forall m, n > N$

Where $\|x_\alpha\|_f = [(x_\alpha|x_\alpha)_f]^{\frac{1}{2}}$

Definition 3.3:

A fuzzy sequence $\{(x_n, \alpha_n)\}$ in afuzzy pre-Hilbert space $(P,(\cdot|\cdot)_f)$ is said to be bounded if there is aconstant $M \geq 0$ *Such that* $\|x_n, \alpha_n\|_f \leq M \quad \forall n$

Definition 3.4:

A fuzzy pre-Hilbert space $(P,(\cdot|\cdot)_f)$ is said to be fuzzy complete if every fuzzy Cauchy equence in P converges,that is,has afuzzy limit which is afuzzy vector of P.

Theorem 3.5:

Every convergent fuzzy sequence is Cauchy.

Proof:

Let $\{(x_n, \alpha_n)\}$ *be afuzzy saquence of fuzzy point in P such that*

$\lim_{n\to\infty} \|(x_n, \alpha_n) - x_\alpha\|_f = 0$

Where $\|x_\alpha\|_f = [(x_\alpha|x_\alpha)_f]^{\frac{1}{2}}$ *and* $\{\alpha_i \in (0,1] | i \in N\}$

Then for every $\varepsilon > 0$, there is an integer $N > 0$

Such that

$\|(x_n, \alpha_n) - x_\alpha\|_F \leq (\varepsilon)/2 \quad \forall n > N$

$$\|(x_m, \alpha_m) - (x_n, \alpha_n)\|_f \leq |(|(x_m, \alpha_m) - x_\alpha + x_\alpha - (x_n, \alpha_n)|)|_f$$

$$\leq \|(x_m, \alpha_m) - x_\alpha\|_f + \|x_\alpha - (x_n, \alpha_n)\|_f$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \varepsilon$$

This shows that $\{(x_n, \alpha_n)\}$ is afuzzy Cauchy sequence.

Proposition 3.6:

Let $(P,(\cdot|\cdot)_f)$ be afuzzy pre-Hilbert space , if $\{(x_n, \alpha_n)\}$ is Converge to x_α and $\{(y_n, \beta_n)\}$ *fuzzy sequnce converges to* y_β *then*

$\{((x_n, \alpha_n)|(y_n, \beta_n))_f\}$ fuzzy sequence converges to ($x_\alpha|y_\beta)_f$

Proof:

$|((x_n, \alpha_n)|(y_n, \beta_n))_f - (x_\alpha|y_\beta)_f| =$

$|((x_n, \alpha_n)|(y_n, \beta_n))_f - ((x_n, \alpha_n)|y_\beta)_f + ((x_n, \alpha_n)|y_\beta)_f - (x_\alpha|y_\beta)_f|$

$\leq |((x_n, \alpha_n)|(y_n, \beta_n) - y_\beta)_f| + |((x_n, \alpha_n) - x_\alpha|y_\beta)_f|$

By Cauchy Schwarz inequality

$|(x_\alpha|y_\beta)_f| \leq \|x_\alpha\|_f \|(y_\beta)\|_f$

$\leq \; \|x_n, \propto_n\|_f \; \|(y_n, \beta_n) - y_\beta\|_f + \|(x_n, \propto_n) - x_\alpha\|_f \; \|y_\beta\|_f \;\; \rightarrow \mathbf{0}$

Hence

$\{((x_n, \propto_n)|(y_n, \beta_n))_f\}$ converges to $(x_\propto|y_\beta)_f$ □

Lemma 3.7:

Every Cauchy fuzzy sequence is bounded.

Proof:

Let $\{(x_n, \propto_n)\}$ be afuzzy Cauchy sequence in afuzzy pre-Hilbert space $(P,(\cdot|\cdot)_f)$

Let N be an index such that

$\|(x_n, \propto_n) - (x_m, \propto_m)\|_f \leq \mathbf{1}$ where m,n≥N

If n≥ N

$\|(x_n, \propto_n)\|_f = \|(x_n, \propto_n) - (x_N, x_N) + (x_N, x_N)\|_f$

$\leq \|(x_n, \propto_n) - (x_N, x_N)\|_f + \|x_N, x_N\|_f \quad by\ (FN3)$

$\leq \quad 1 + \|x_N, x_N\|_f$

Thus if M is the largest of the number $1 + \|x_N, x_N\|_f, \|x_1, \propto_1\|_f,$
$\|(x_2, x_2)\|_f \;, \ldots \; \|(x_{N-1}, x_{N-1})\|_f$ then one has $\|(x_n, \propto_n)\|_f \leq M \quad \forall n$

Thus $\{(x_n, \propto_n)\}$ is bounded.

Theorem 3.8:

If $\{(x_n, \propto_n)\}$ and $\{(y_n, \beta_n)\}$ be Cauchy fuzzy sequences in $(P,(\cdot|\cdot)_f)$ then $\{((x_n, \propto_n)|(y_n, \beta_n))_f\}$ is cauchy fuzzy sequence.

Proof:

For all n we have

$((x_n|\propto_n)|(y_n, \beta_n))_f - ((x_m, \propto_m)|(y_m, \beta_m))_f$

$= ((x_n, \propto_n) - (x_m, \propto_m)|(y_n, \beta_n) - (y_m, \beta_m))_f +$

$((x_n, \propto_n) - (x_m, \propto_m)|(y_m, \beta_m))_f + (x_m, \propto_m)|(y_n, \beta_n) - (y_m, \beta_m))_f$

$\leq \|(x_n, \propto_n) - (x_m, \propto_m)\|_f \|(y_n, \beta_n) - (y_m, \beta_m)\|_f + \|(x_n, \propto_n) - (x_m, \propto_m)\|_f$

$\|(x_n, \propto_n) - (x_m, \propto_m)\|_f \|(y_m, \beta_m)\|_f + \|(x_m, \propto_m)\|_f \|(y_n, \beta_n) - (y_m, \beta_m)\|_f$

$\{(x_n, \propto_n)\}$ is bounded [by lemma 3.7] thus the right side→0 as m,n→∞

Definition 3.9:

Afuzzy pre-Hilbert space $(P,(\cdot|\cdot)_f)$ is said to be fuzzy Hilbert space if it is fuzzy complete with respect to the fuzzy normed $\|x_\propto\|_f = [(x_\propto|x_\propto)_f]^{\frac{1}{2}}$ $where\ x_\propto \in P$

Proposition 3.10:

If $\{x_n\}$ *is asequence in the ordinary Hilbert space* $(P,(\cdot|\cdot))$ *converge* **to x then** $\{(x_n, \propto_n)\}$ is asequence of fuzzy points in the fuzzy Hilbert space **$(P,(\cdot|\cdot)_f)$ converge to $x_\propto$ where** $\propto_1, \propto_2, \ldots \in (0,1]$

Proof:

$x_n \rightarrow x\ in\ P$, then $\lim_{n\rightarrow\infty} \|x_n - x\| = 0.$

Take sequence $\propto_n \in (0,1]$ converge to $\propto$ [this is possible $\propto_n = \left(1 - \frac{1}{n}\right)\propto$]

Hence $\{(x_n,\propto_n)\}$ is afuzzy *sequence in* $(P,(\cdot|\cdot))$.

Now since $\lim_{\tau}(n\to\infty)$ 〖||x_n − *x*|| 〗_*f* (λ) = 〖$\lim_{\tau}(n\to\infty)$ 1/λ||〗〖x_*n* − *x*〗||〗_*f* = 0〗〗 where λ=min

{∝, ∝ _n} thus $\{(x_n,\propto_n)\}$ is fuzzy sequence in the fuzzy Hilbert space$(P,(\cdot|\cdot)_f)$.

Proposition 3.11:

If $\{x_n\}$ *is* cauchy *sequence* of fuzzy points in **an** ordinary Hilbert space **(P,(. | .)) then** $\{(x_n,\propto_n)\}$ **is acauchy sequence of afuzzy point in the fuzzy Hilbert space** $(P,(\cdot|\cdot)_f)$**.**

Proof:

is Cauchy sequence in P.$\{x_n\}$

i.e $\forall\varepsilon>0, \exists M>0$ such *that*

$||x\_m - x\_n|| < \varepsilon \quad \forall m,n > M$

Take $\propto_n$ be *a* sequence w*here* $\propto_n\in(0,1]$ w*hich is cauchy sequence*

Where [$\propto\_n = (1-1/n)\propto$]

Hence $\{(x_n,\propto_n)\}$ is afuzzy *sequence in* $(P,(\cdot|\cdot))$.

Since |〖||x_*m* − *x*_n ||〗_*f* (λ) = 1/*λ* |(|x_*m* − x_*n* |)| < ε

i.e $\{(x_n,\propto_n)\}$ *is* a fuzzy *sequence* in *a fuzzy* Hilbert *space*$(P,(\cdot|\cdot)_f)$.

Theorem 3.12:

If $(P,(\cdot|\cdot))$is an ordinary Hilbert space *then*$(P,(\cdot|\cdot)_f)$ **is afuzzy Hilbert space by defining**

$(x_\propto|y_\beta)_f = \frac{1}{\lambda}(x|y) \quad \lambda = \min\{\propto,\beta\}\ \alpha,\beta\in(0,1]$

Proof:

(P,(.|.)_f) is afuzzy pre-Hilbert space by proposition (2.2)

Since (P,||. ||) is complete normed space where ||x|| =√((x|x)*s*) so (P, 〖|| · ||〗_f) is

A fuzzy complete normed space where $||x_\propto||_f = \frac{1}{\propto}||x|| \quad \forall x_\propto \in P$

Thus $(P,(\cdot|\cdot)_f)$*is a* fuzzy Hilbert space.

Example 3.13:

The space $L^2[a,b]$*is a* Hilbert *space* with *pre* − Hilbert *defined by*

$(x_\propto|y_\beta)_f = \int_a^b x(t)\,y(t)\,dt \qquad$ where $x(t), y(t) \in L^2[a,b]$

Hence (L^2 [a, *b*] ,(· | ·)_*f*) *is* fuzzy Hilbert space *with* fuzzy pre − *Hilbert*

Defined by $(x_\propto|y_\beta)_f = \frac{1}{\lambda}(x|y)$.

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بعض النتائج على فضاءات ما قبل هلبرت الضبابية

رغد ابراهيم صبري

قسم العلوم التطبيقية

الجامعة التكنولوجية

الخلاصة:

هدف البحث هو تقديم تعريف جديد لفضاءات ماقبل هلبرت الضبابية(فضاء الضرب الداخلي الضبابي) ومناقشة العلاقة بين فضاءات ماقبل هلبرت الضبابية و الاعتيادية .

أعطينا اهتماماً كبيراً لمفهوم تقارب المتتابعات (مثلاً متتابعة كوشي التي عناصرها نقاط ضبابية)ومفهوم الحدودية لتلك المتتابعات.