Spectral Methods to Investigation the Stability for Parabolic Problems

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ABSTRACT

In this paper, the formula of second order parabolic problems with finite difference method (mesh method) has been presented and we used the spectral method to investigate the stability for parabolic problems.

1- INTRODUCTION

Spectral methods have been used extensively during the last decades for numerical solution of partial differential equations. The expression spectral methods has different meanings for several subareas of mathematics, like functional analysis and signal processing. Jeseviciute [1], explained the stability of implicit difference scheme for parabolic equation subject to integral conditions, he investigated the stability of difference scheme for parabolic equations subject to nonlocal integral conditions. Hanif, Hans and Alaeddin [2], they considered the exponential stability and approximate controllability, dual phase lagging equation and this equation is linear, time independent partial differential equation modeling the heat distribution in a thin film. Hannes [3], the spectral method has been explained as Fourier modes which use the representation of the solution by orthogonal eigenfunctions of some linear partial differential operator. In our work it has the meaning of high accuracy numerical method to solve partial differential equations. In this paper we investigate the stability for parabolic problems in some cases not all, presented a finite- difference schemes by using spectral method. In fact, we have to make investigations of the stability of schemes because of these schemes don't work for arbitrary choice of important conditions to satisfies the stability. The simplest way to express about the meaning of the stability in numerical methods is : first, we say in a given scheme is stable if it satisfies such conditions that the error of approximation is decreasing so in the scheme which presented in our

paper there is some conditions with the steps h and τ , and second the scheme is not stable this means when we calculate all layers and this errors accumulates and gets bigger and bigger, in this case we have a wrong solution. We know that choosing the smaller step gives us more precise approximation in general case, but if the scheme is not stable or unstable this is not enough doesn't work. Third we say that the scheme is absolutely stable if for an arbitrary choice of steps h and τ we have a stable scheme. So here we gave in idea about the meaning of stability in our case. In fact, we use the scheme for parabolic problems which describe an approximate solution by using mesh method to obtain a solution and considering the sufficient condition of stability for one of the cases and the scheme presented as weighted scheme.

2. Difference scheme for parabolic problems.

Differential Problem

Consider the following boundary value problem for heat equation

$$u_t = u_{xx} + f(x, t)$$

…(2.1) [4]

$$\text{In } G = \{(x,t): 0 < x < 1, \quad 0 < t < T\}$$

The Initial Condition

$$u\big|_{t=0} = \varphi(x), \qquad 0 \le x \le 1,$$

The Boundary Conditions

$$u\big|_{x=0} = \mu_1(t), \qquad 0 \le t \le T,$$
$$u\big|_{x=1} = \mu_2(t), \qquad 0 \le t \le T .$$

And the Compatibility Conditions

$$\varphi(0) = \mu_1(0), \varphi(1) = \mu_2(0)$$

We assume that the compatibility conditions $\varphi(0) = \mu_1(0), \varphi(1) = \mu_2(0)$ hold and $f(x,t) \in C(\bar{G}), \varphi(x) \in C([0,1]), \mu_2(t) \in C([0,T]).$

Then the problem has unique solution from the class $[C]^2 (G) \cap C(\bar{G})$.

Equation (2.1) describes the heat distribution in a thin road of length 1 for given initial heat distribution $\varphi(x)$ and for given lows $\mu_1(t), \mu_2(t)$ of changing the temperature at the two ends of the road. The variable x is a space variable , the variable t has a sense of time.

Difference Scheme

We construct difference scheme to solve the problem approximately.

In the region G we introduce the mesh $w_{h\tau}.$

$$w_{h\tau} = w_h \times w_\tau, \quad where: w_h = \left\{x_i = ih, \quad h = \frac{1}{n}, i = 0, 1, …, n\right\}$$

$$w_\tau = \left\{t_j = j\tau, \quad \tau = \frac{T}{m}, j = 0, 1, …, m\right\}$$

We approximate the derivative u_t of equation in the following way:

$$u_t(x_i, t_j) = \frac{u_{i,j+1} - u_{i,j}}{\tau} + \frac{\tau}{2}u_{tt}(x_i, \gamma)$$
$$= \frac{u_{i,j+1} - u_{i,j}}{\tau} + o(\tau), \qquad t_j \le \gamma \le t_{j+1}.$$
…(2.2)

And for the derivative u_{xx} we use the following approximation:

$$u_{xx}(x_i, t_j) = \frac{u_{i,+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{h^2}{12}u_{xxx}(\delta, t_i)$$
$$= \frac{u_{i,+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + o(h^2), \qquad x_{i-1} \le \delta \le x_{i+1}.$$
…(2.3) [5]

Next, we substitute the problem (2.1) with the difference equation of the form:

$$\frac{u_{i,j+1} - u_{i,j}}{\tau} = (1 - \sigma)\frac{(u_{i+1,j} - 2u_{i,j}| + u_{i-1,j})}{h^2} + \sigma\frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2} + f_{i,j},$$
$$i = 1, …, n, \qquad j = 1, …, m - 1$$
….(2.4) [6]

$$u_{i,0} = \varphi_i, \qquad i = 0, …, n$$
…(2.5)

$$u_{0,j} = \mu_{1j}, \qquad u_{n,j} = \mu_{2j}, \qquad j = 1 …, m$$
…(2.6)

Where the fixed number $\sigma \in [0, 1]$, we call weight or weighted scheme. We will study the case when the parameter $\sigma = 0$, we will have explicit scheme.

We will use a technique called spectral method which gives a possibility to find conditions for stability with respect to initial data for homogenous linear difference schemes of constant coefficients.

Let us seek for particular solutions of homogeneous difference scheme of the kind:

$$y_{kj}(\theta) = q_j(\theta)e^{ikh\theta}, k = 0, 1, …, n, \qquad j = 0, 1, …, m$$
… (2.7)

Where i is the imaginary part, θ is appropriate real number.

For $j = 0$, we find $y_{k0} = e^{ikh\theta}$, thus y_{k0} is bounded for every θ and k=0,1,…,n. Then the solutions of kind (2.7) would be stable with respect to the initial data, if

$$|y_{kj}(\theta)| \le |y_{k0}(\theta)|, \qquad k = 0, 1, …, n, \qquad j = 0, 1, …, m.$$ and it is enough that

$$|q(\theta)| \le 1, \qquad \forall\theta.$$

Now, Let us applied the spectral method to the weighted scheme (1.4) written as:

$$y_t(i, j) = \sigma y_{xx}(i, j + 1) + (1 - \sigma)y_{xx}(i, j)$$
$$= f_{h\tau}(i, j), \qquad i = 1, …, n - 1, \qquad j = 0, 1, …, m - 1.$$

When $\sigma = 0$, we get

$$y_t(k, j) - y_{xx}(k, j) = f_{h\tau}(k, j)$$

Or

$$y_t(k, j) - y_{xx}(k, j) \quad with \quad y_{kj}(\theta) = q_j(\theta)e^{ikh\theta}$$
$$\frac{y_{kj+1} - y_{kj}}{\tau} = \frac{y_{k+1j} - 2y_{kj} + y_{k-1,j}}{h^2} \quad and \quad y_{kj}(\theta) = q_j(\theta)e^{ikh\theta}$$
$$\frac{q_{j+1}e^{ikh\theta} - q_j e^{ikh\theta}}{\tau} = \frac{q_j e^{i(k+1)h\theta} - 2q_j e^{ikh\theta} + q_j e^{i(k-1)h\theta}}{h^2}$$

$$\frac{q_j(q-1)}{\tau}e^{ikh\theta} = \frac{q_j(e^{ih\theta}-2+e^{-ih\theta})}{h^2}e^{ikh\theta}$$

$$\frac{q-1}{\tau} = \frac{e^{ih\theta}+e^{-ih\theta}-2}{h^2}$$

$$\frac{q-1}{\tau} = \frac{2\cos\theta h-2}{h^2}$$

$$\frac{q-1}{\tau} = \frac{2\left(-2\sin^2\frac{\theta h}{2}\right)}{h^2}$$

$$q = 1 - \frac{4\tau}{h^2}\sin^2\frac{\theta h}{2}$$

But $|q| \le 1,$ enough to obtain stability.

$$|1 - 4\tau/h^2\ \sin^2[\theta h/2]| \le 1$$

$$-1 \le 1 - \frac{4\tau}{h^2}\sin^2\frac{\theta h}{2} \le 1$$

$$-2 \le 1 - \frac{4\tau}{h^2}\sin^2\frac{\theta h}{2} \le 0$$

$$-2 \le \frac{-4\tau}{h^2} \le 0$$

$$\frac{1}{2} \ge \frac{\tau}{h^2} \qquad\qquad \text{…(2.8)}$$

Then $\tau \le \frac{h^2}{2}.$ This is the condition of stability.

Hence, the explicit difference scheme is conditionally stable ,it means the scheme is stable when the condition (2.8) is fulfilled.

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المستخلص

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