On The Weak and Rather Weak Homotopy - Extension Property

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The aim of this paper is to compare the RWHEP (Theorem 3.2), we define and study several generalizations of homotopy equivalence maps.

1.INTRODUCTION

The weak homotopy extension property (WHEP) of a pair of spaces (X, A) is very important in problems involving relative homotopy equivalence (see for instance, Dold's theorem [7], 2.18).

R. Brown, in ([1] pp. 255) considers weaker concept , the rather weak HEP (RWHEP) which is interesting for its applications in the fibrations of groupoids.

The purpose of this paper is to compare the WHEP and RWHEP . Throughout this paper we work in the category of topological spaces and continuous maps .

(for definitions not given here see [3]).

2 . PRELIMINARIES

Definition2.1 [3]: Let $i : A \to X$ and $i' : A \to Y$ be two maps, a map $f : X \to Y$ is called a map under A if $f \circ i = i'$.

Two maps $f,g : X \to Y$ are called homotopic under A, written $f \overset{A}{\simeq} g$ if there exists a homotopy $H : X \times I \to Y$ such that $H_t : X \to Y$ is a map under A where $H_t(x) = (x,t)$.

Definition2.2 [1]: A map $i : A \to X$ is said to have the (WHEP) with respect to a space Y if given maps $f : X \to Y$ and $H : A \times I \to Y$ such that

$H(a, 0) = f(i(a))$ for all $a \in A$, then there exists a homotopy $F : X \times I \to Y$ such that $F \circ (i \times 1_I) = H$ and $F_0 \overset{A}{\simeq} f$.

Definition2.3 [1]: A map $i : A \to X$ is said to have the (RWHEP) with respect to a space Y if given maps $f : X \to Y$ and $H : A \times I \to Y$ such that $H(a, 0) = f(i(a))$ for all $a \in A$, then there exists a homotopy $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ for all $x \in X$ and $F \circ (i \times 1_I) \simeq H$ rel. $A \times I$.

If i has the WHEP (RWHEP) with respect to every space , we say that i has the WHEP (RWHEP) with respect to every space, we say that i has the WHEP (RWHEP) or i is a W cofibration (RW cofibration).

Definition2.4 [7]: Let $i : A \to X$ be a map, then $f, g : X \to Y$ such that $f \circ i = g \circ i$ are said to be homotopic under i, denoted by $f \overset{i}{\simeq} g$ if there is a homotopy $H : f \simeq g$ such that $H \circ (i \times 1_I) \simeq O_{foi}$.

Definition2.5: Let $i : A \to X$ be a map, then $f, g : X \to Y$ such that $f \circ i = g \circ i$ are said to be rather weakly homotopic under i denoted by $f \overset{(i)}{\simeq} g$ if there is a homotopy

$H : f \simeq g$ such that $H \circ (i \times 1_I) \simeq O_{foi}$ rel. $A \times I$.

Remark2.6 : Note that $\overset{i}{\simeq}$ and $\overset{(i)}{\simeq}$ are equivalence relations in the set of maps $X \to Y$. The homotopy classes under i represented by f are denoted by $[f]^i$ and $[f]^{(i)}$.

Definition 2.7[8]: Let $i : A \to X$, $i' : A \to Y$ be maps, then $f : X \to Y$ such that $f \circ i = i'$ is said to be cofiber homotopy equivalence under i, denoted by $f : X \overset{i}{\simeq} Y$, if there is a map $g : Y \to X$ such that $g \circ i' = i$, $f \circ g \overset{i}{\simeq} 1_Y$ and $g \circ f \overset{i}{\simeq} 1_X$.

Definition 2.8: Let $i : A \to X$, $i' : A \to Y$ be maps, then $f : X \to Y$ such that $f \circ i = i'$ is said to be RW cofiber homotopy equivalence under i, denoted by $f : X \overset{(i)}{\cong} Y$, if there is a map $g : Y \to X$ such that $g \circ i' = i$, $f \circ g \overset{(i)}{\cong} 1_Y$ and $g \circ f \overset{(i)}{\cong} 1_X$.

Proposition2.9 : Let $i : A \to X$ be a map, let $f, g : X \to Y$ be two maps such that $f \circ i = g \circ i$ then $f \overset{A}{\cong} g$ if and only if $f \overset{i}{\cong} g$.

Proof : Let $f \overset{A}{\cong} g$ then there is $H : f \cong g$ such that $H_t(i(a)) = f(i(a)) = g(i(a))$, thus $H \circ (i \times 1_I) = O_{f \circ i}$ hence $f \overset{i}{\cong} g$. The converse is obvious ♦

The following example shows two maps which are homotopic under i :

Example2 .10: Let $i : I \to R^+$ be such that $i(a) = 2a$. Define $f, g : R^+ \to R$ as follows :

$$\mathbf{f}(\mathbf{x}) = \begin{cases} \mathbf{x} & 0 < \mathbf{x} \le 2 \\ 2 & \mathbf{x} > 2 \end{cases}$$

$$\mathbf{g}(\mathbf{x}) = \begin{cases} \mathbf{x} & 0 < \mathbf{x} \le 2 \\ 4 - \mathbf{x} & \mathbf{x} > 2 \end{cases}$$

then $f \circ i = g \circ i$. To show that $f \overset{i}{\cong} g$, we have that $f \cong g$ since any two maps $f, g : X \to R$ are homotopic, thus there is $H : f \cong g$, $H(x, t) = (1 - t) f(x) + t g(x)$. Then $H_t \circ i = f \circ i$. Hence $f \overset{i}{\cong} g$.

3. SOME RESULTS

In this section we give a comparison theorem (3.2) of RWHEP and WHEP which is analoguos to theorem (4.2) of [4]. We need the following lemma which is the dualization of lemma (1.5) in [5], its proof is also dual.

Lemma 3 .1: If $i : A \to X$ has the RWHEP and $g : X \to X$ satisfies $g \circ i = i$ and $g \cong 1_x$, then there exists a map $g' : X \to X$ such that $g' \circ g \overset{(i)}{\cong} 1_x$.

Theorem 3.2: Let $i : A \to X$ be a map which has the RWHEP . Then the following are equivalent :

(1) i has the WHEP .

(2) given a map $f : A \to Y$, if $f' : X \to Y$ is such that $f' i = f$, then $[f']^{(i)} = [f']^{i}$.

(3) for every map $g : X \to X$ such that $g\ i = i$, then $[g]^{(i)} = [g]^{i}$.

(4) for every map $g : X \to X$ such that $g\ i = i$ and $g \cong 1_x$, there exists a map $g' : X \to X$ such that $g' g \overset{i}{\cong} 1_X$.

Proof: (1) $\Rightarrow$ (2). To show that $[f]^{i} \subset [f]^{(i)}$; let $g' \in [f']^{i}$, then $g' \overset{i}{\cong} f'$, so there is $H : g' \cong f'$, $H(i(a), t) = f\, i(a) = O_{f \circ i}(a, t)$. So $g' \overset{i}{\cong} f'$, Hence $g' \in [f']^{(i)}$.

To show that $[f']^{(i)} \subset [f']^{i}$; Let $g' \in [f']^{(i)}$, hence there are homotopies

$G : g' \cong f'$ and $F : G(i \times 1_I) \cong O_{goi}$ rel. $A \times I$ .Define a homotopy $\bar{F} : A \times I \times I \to Y$ by :

$$\bar{F}(a, t, s) = \begin{cases} G(i(a), s) & 0 \le t \le 1/2 \\ & \qquad s, t \in I \\ F(a, s, 2t-1) & 1/2 \le t \le 1 \end{cases}$$

Since $i : A \to X$ has the WHEP, so $i \times 1_I : A \times I \to X \times I$ has WHEP hence there exists a homotopy extension $\Phi : X \times I \times I \to Y$ such that $\Phi(x, t, 0) = G(x, t)$, $\quad x \in X$, $t \in I$

$$\Phi \circ (i \times 1_{I \times I}) = \bar{F} : A \times I \times I \to Y$$

Define a homotopy $H : X \times I \to Y$ by :

$$H(x, t) = \begin{cases} \Phi(x, 0, 3t) & 0 \le t \le 1/3 \\ \Phi(x, 3t-1, 1) & 1/3 \le t \le 2/3 \quad , \quad x \in X \\ \Phi(x, 1, 3-3t) & 2/3 \le t \le 1 \end{cases}$$

Then $F : g' \cong f'$ and $F(i(a), t) = f(a)$, $a \in A$, $t \in I$. That is $g' \overset{i}{\cong} f'$, $g' \in [f']^{i}$ and hence $[f']^{(i)} \subset [f']^{i}$. Therefore $[f']^{(i)} = [f']^{i}$.

(2) $\Rightarrow$ (3), as $g \circ i = i$ so by taking $Y = X$, $g = f'$ we have $[g]^{(i)} = [g]^{i}$

(3) $\Rightarrow$ (4) $\Rightarrow$ (1) These are dual to that of (3.1) in [5] . ♦

We have the following result :

Corollary3.3: If $i : A \to X$ and $i' : A \to X'$ both have the WHEP and $f : X' \to X$ is such that $f\, i' \cong i$, $f : X \cong X'$, then f is a homotopy equivalence under i, thus $f : X \overset{i}{\cong} X'$.

Proof: Since $f : X \cong X'$, so there is $f' : X' \to X$ such that $f' \circ f \cong 1_X$, $f \circ f' \cong 1_{X'}$.

Let $F : f' \circ f \cong 1_X$, define a homotopy $\Phi : A \times I \to X$ by :

$$\Phi(a,t) = \begin{cases} \mathbf{f'(a)} & 0 \le \mathbf{t} \le 1/2 \\ \mathbf{F(a}, 2\mathbf{t}-1) & 1/2 \le \mathbf{t} \le 1 \end{cases} , \ \mathbf{a} \in \mathbf{A}$$

Then Φ and f' have a homotopy extension $G : X' \times I \to X$ such that $G(x',0)=f'(x'), x' \in X'$, $G|_{A \times I} = \Phi$. Define $f'' : X' \to X$ by $f''(X') = G(X', 1)$, $x' \in X'$, then $f''(a)=a$, $a \in A$ and $f'' \cong f'$. Hence $f'' f \cong f' f \cong 1_X$, so by (3.2 (4)), there is a map $g : X \to X$ such that $g f'' f \overset{i}{\cong} 1_X$. Let $g' = g f' : X' \to X$, then $g' f \overset{i}{\cong} 1_X$ and $g \cong g f' f \cong 1_X$. Thus $f g' = f g f'' \cong f f'' \cong 1_{X'}$. Hence $g' : X' \cong X$.

Then we can apply the same argument to g' and find $g'' : X \to X'$, such that $g'' g \overset{i}{\cong} 1_{X'}$ and $f g' \overset{i}{\cong} g'' \; g' f g' \overset{i'}{\cong} g'' g' \overset{i'}{\cong} 1_{X'}$.

It is known [8] that if $i : A \to X$ and $i' : A \to X'$ have the WHEP and $f : X \to X'$ is a homotopy equivalence such that $f \circ i = i'$, then f is a homotopy equivalence under A.

To prove the RW version of this result, the following lemma is needed :

Lemma3.4: Let $i : A \to X$ and $i' : A \to X'$ be maps and $f : X \to X'$ be a map such that $f \circ i \cong i'$. If i has the RWHEP then $f \overset{i}{\cong} g$ for some $g : X \to X'$ such that $g \circ i = i'$.

Proof: Since $f \circ i \cong i'$, so there is $H : A \times I \to X'$ such that $H_0 = f \circ i$ and $H_1 = i'$ since i has the RWHEP, there is $F : X \times I \to X'$ such that $F_0 = f$ and

$F \circ (i \times 1_I) \cong H$ rel $A \times \dot{I}$, Let $F_1 = g$ then $g = F_1 \cong F_0 = f$ and $g \circ i = F_1 \circ i = H_1 = i'$.

Note that $H_1(a) = H(a, 1)$ and $F_1(x) = F(x, 1)$ in the above proof.

Theorem3.5: If $i : A \to X$ and $i' : A \to X'$ have the RWHEP and $f : X \to X'$ is a map such that $f \circ i = i'$, $f : X \cong X'$, then f is a RW cofiber homotopy equivalence, $f : X \overset{(i)}{\cong} X'$.

Proof: As $f : X \cong X'$, there is a homotopy inverse $h : X' \to X$ of f. Then

$h i' = h \circ f \circ i \cong i$. Hence by Lemma (3.4), $h \cong h'$ for some $h' : X' \to X$ such that $h' i' = i$. Since $h' \circ f \cong h \circ f \cong 1x$, and as $h' \circ f \circ i = h' \circ i' = i$, so by Lemma (3.1), there exists a map $h'' : X \to X$ such that $h'' \circ i = i$ and $h'' \circ h' \circ f \overset{(i)}{\cong} 1x$.

Thus f has a RW homotopy left inverse $g = h'' \circ h'$ such that $g \circ i' = i$, then g is a homotopy equivalence, since f is a homotopy equivalence and so the same argument applied to g instead of f , shows that g has a rather weakly homotopy left inverse g' such that $g' \circ i = i'$. Thus g has both a RW homotopy inverse f and a RW homotopy left inverse g'. Hence g is a RW cofiber homotopy equivalence . So f is a RW cofiber homotopy equivalence.

Remark3.6: The above proof can be given as a dualization of Remark 1.6 in [5].

Corollary3.7: If $i : A \to X$ has a RWHEP, then there is a map $i' : A \to X'$ has HEP and a map $f : X' \to X$ which is a RW homotopy equivalence such that $f \circ i' = i$. (i.e f is a RW cofiber homotopy equivalence)

Proof: Since i has the RWHEP , then i can be factored as $i = f \circ i'$ where i' has a HEP and f is a homotopy equivalence by ([7] , prop.2) . Hence f is a RW homotopy equivalence by (3.5).

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