

Using Linear programming to find solution volterra Integral Equation.

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الخلاصة :

تعتمد هذه الطريقة على تقريب الحل باستخدام تقنية جديدة تعتمد على طريقة البرمجة الخطية باستخدام طريقة (Simplex method) لإيجاد الحل الأفضل لهذه المعادلات عدديا .

List of Abbreviations

| Symbol | Definition |
|---------------------|---|
| VIE | volterra Integral Equation . |
| VIE 1 st | volterra Integral Equation of first kind. |
| ODE | Ordinary differential Equation. |
| $S^{-1}\{F\}$ | Laplace transform of f. |
| $S^{-1}\{F\}$ | Inverse Laplace transform of f. |
| LP | Linear programming. |
| L.S.E | Least square error. |
| L.S.E g | Least square error of g. |
| R.T | Running time. |

The Aim of these studies:

The major purpose of this thesis is:

- 1- To survey as well as to modify L.P. method have been used solve volterra integral equation of convolution type.
- 2- To write successful program for this method.

Preface

Considers the use of linear method as an approximate method for treating these equations.

For this method a. program is written, example are solved, results are tabulated, and the computer programs is written in turbo Pascal (ver.7)

Section (1)

1 – 1 Classification of linear integral equations

Before discussing the appearance of the integral equations.

It will be necessary present some definitions and introduce a preliminary classification of linear Integral equation. {2}

Def (1-1)

The general linear integral equations can be represented as:

$$h(x)f(x) = g(x) + \lambda \int_a^b k(x, y)f(y)dy.....(1-1)$$

$h(x)$, $g(x)$: are known equation of x

$f(x)$: un known equation.

$k(x,y)$: is called of the integral equation .

λ : Is called the characteristic value.

(In this work we take $\lambda =1$)

Def (1-2)

The integral equation (1-1) is called an integral equation of the first kind when ^{1}

$h(x) = 0$, i.e.

$$g(x) = \int_a^b k(x, y)f(y)dy.....(1-2)$$

Def (1-3)

(a) -The integral equation in (1-1) is called an integral equation of the second kind $h(x) =1$, which is:

$$f(x) = g(x) = \int_a^b k(x, y)f(y)dy.....(1-3)$$

(b) -If a. and b are constants then equation (1-2) and (1-3) are called (Fredholm integral equation) of the first and second kind respectively.

(c) - The integral equation (1-2) and (1-3) are called a (volterra integral equation) when $b=x$, $a=0$ hence the integral equations.

$$g(x) = \int_0^x k(x, y) f(y) dy \dots\dots\dots(1-4)$$

$$f(x) = g(x) + \int_0^x k(x, y) dy \dots\dots\dots(1-5)$$

Respectively volterra integral equation of the first and second kind respectively. ^{5}

Def (1-4)

If the kernel in equation (1-1) depends only on the difference $(x-y)$ such a kernel is called a (difference kernel) and the equation (1-1) with this kind of kernel B called an (integral equation of convolution type) ^{3}

Def (1-5)

The kernel $k(x, y)$ is called separable or degenerate if it is of the form

$$k(x, y) = \sum_{k=1}^n a_k(x)b_k(y).....(1-7)$$

Where n is finite.

Notation:

This work is concerned with volterra integral of convolution type of first and second kind

$$g(x) = \int_0^x k(x-y)f(y)dy.....(1-8)$$

$$f(x) = g(x) + \int_0^x k(x,y)f(y)dy.....(1-9)$$

Section (2)

Methods of linear program.

(2-1) Simplex Method

The principle algorithm used in solving linear program is simplex method, this method deals with finding extreme values of linear function when the variable are constrained by inequalities.

Simplex method can be summarized by the following .^{2}

Step 1:

Set up the initial tableau.

$$\sum_{j=1}^n \alpha_{j=1} \alpha_j = b \dots \dots \dots \text{for} \dots \dots \dots i = 1, \dots \dots \dots, m$$

Step 2 :

Finding which element should enter the basis by choosing the most positive value form the objective function ($z_j - c_j$)

Step 3 :

Choose the minimum value of ratio $(\frac{b_i}{a_{is}})$ such that $a_{is} > 0$ to

determine which element should leave the basis .

Step 4 :

Repeat the steps (2 and 3) until all the values of $(z_j - c_j) \leq 0$ to test optimality , if $c_j \alpha_j \leq 0$ them c_j s satisfy the system of LP simplex method can be explained in the table below :

Initial table:

| Basis | $\alpha_1 \alpha_2 \dots \alpha_n \dots \alpha_{n+1} \dots \alpha_{n+m}$ | | | | | | | Soln. | Row |
|----------------|--|----------|----------|-------|---|-------|---|-------|-----------|
| α_{n+1} | a_{11} | a_{12} | a_{1n} | 1 | 0 | 0 | 0 | b_1 | R_1 |
| α_{n+2} | a_{21} | a_{22} | a_{2n} | 0 | 1 | | | b_2 | R_1 |
| . | . | . | . | | | | . | . | . |
| . | . | . | . | | | | . | . | . |
| . | . | . | . | | | | . | . | . |
| . | . | . | . | | | | . | . | . |
| α_{n+m} | a_{m1} | a_{m2} | a_{mn} | 0 | 0 | | 1 | b_m | R_m |
| Z | C_1 | C_2 | | C_n | 0 | 0 | 0 | C_0 | R_{m+1} |

Transform table

| Basis | α_1 | α_n | α_n | α_{n+m} | | |
|----------------|---|------------|--------------------------------------|----------------|---|---|
| α_{n+1} | $a_{11} - \frac{a_{15}}{a_{rs}} a_{r1}$ | \dots | $a_{1n} - \frac{a_{15}}{a_{rs}}$ | 1 | 0 | $b_1 \frac{a_{15}}{a_{rs}} b_r$ $b_m - \frac{a_{15}}{a_{rs}} b_r$ |
| Z | $c_1 - \frac{c_s}{a_{rs}} a_{r1}$ | 0 | $c_n - \frac{a_{ms}}{a_{rs}} a_{rm}$ | | | $c_0 - \frac{c_s}{a_{rs}} b$ |

transformed..... $\overline{R} = \overline{R}_1 - R_1 - \frac{a_{15}}{a_{rs}} R$

$\overline{R}_m = R_m - \frac{a_{ms}}{a_{rs}} R_r$

and..Intgeral.... $\overline{R}_j = R_j - \frac{a_{js}}{a_{rs}} R_r, \dots, j = 1, \dots, r + 1, m$

$\overline{R}_r = \frac{R_r}{a_{rs}}$

$\overline{R}_{m+1} = R_{m+1} - \frac{c_s}{a_{rs}} R_s$

(2 –2) Dual Simplex method

Dual Simplex method is helpful in finding the solution of linear programming problem a number of different right hand side actors bi..

It is also used when new constraints are added to an LP problem for which the optimal solution has already been obtained. {6}

The following advantages can be obtained by using dual property 1-If the primal problem continuous are large number of rows

(Constraints) and a smaller number of columns (variables), the computational procedure can be considerably reduced be converting into dual and them solving.

2- Duality in linear programming has certain for reaching consequences of economic nature.

3- Calculation of the dual cheeks the accuracy of prime solution. In general if the primal problem.

$$\text{Minimize } z = \sum_{j=1}^n c_j \alpha_j \text{ Subject to :}$$

$$\sum_{j=1}^n a_{ij} \alpha_j \geq b_i \dots\dots \text{for} \dots\dots i = 1, \dots, m$$

Them the dual problem is given by

Minimize $w = \sum_{i=1}^m b_i y_i$ Subject to:

$$\sum_{i=1}^m a_{ij} y_i \leq c_j \dots \text{for } i = 1, \dots, m$$

TO above transform is valid only for symmetric problem.

In order to convert the primal problem to the dual problem, we these steps:

Step 1

Convert the problem in to maximize problem it is initially in for.

Step 2

Convert type constraints, if any, in to type by multiplying both sides of such constraints by -1 .

Step 3

Convert the inequality constants in to equality by the addition of slack variables and obtaining basic solution.

Step 4

To choose the basic, the most negative value form the objective function in selected.

Step 5

Choos the minimum value of ratio ($\frac{c_j}{a_{js}}$) such that $a_{js} >$

0 to leave basic .

Step 6

Repeat step (4 and 5) until all the value of objective function be ≥ 0 .

Section (3)
Technique for solving VIT using L.P

Now in this section linear programming method is used to approximate the solution of volterra integral equation.

(3-1) The solution of VIT ^{2nd} using L.P

Linear programming method can be used to approximate the solution of equation (1-9)

$$f(x) = g(x) + \sum_0^x k(x-y)f(y)dy$$

or on the operator notation ,we have

$$f = g + Lf \dots \dots \dots (3-1)$$

This equation can be solved approximate for f by considering the approximate solution to be

$$f_n = \sum_{j=1}^n \alpha_j \phi_j \dots \dots \dots (3-2)$$

Where $(\phi_1 \dots \phi_n)$ are chosen basis function $\phi_j = X^j$ and α_j being

Undetermined constant coefficients. Substituting equation (4-1) yields

$$f_n = g + Lf_n$$

$$\sum_{j=1}^n \alpha_j \phi_j = g + \sum_{j=1}^n \alpha_j (L\phi_j) + \delta_n$$

Where δ_n is the residual function

so

$$\delta_n = g - \left[\sum_{j=1}^n \alpha_j \phi_j - \sum_{j=1}^n \alpha_j \phi_j (L\phi_j) \right]$$

$$\delta_n = g - \sum_{j=1}^n (1-L)\alpha_j \phi_j = g - \sum_{j=1}^n p\alpha_j \phi_j$$

Then

$$\delta_n = g - \sum_{j=1}^n \alpha_j \psi_j \dots \dots \dots (3-3)$$

Where

$$\psi_j = p\phi_j = (1-L)\phi_j \dots \dots \dots \text{equivalently} \dots \dots \dots (6-3)$$

Quadrature rules (trapezoidal rule and Simpson 1/3 rule) are used to estimate the ψ_j^S rules numerically .

Minimizing the residual $\delta_n(x)$ in equation (3-2)

$$\|\delta\| = \text{Min} \left\| g - \sum_{j=1}^n \alpha_j \psi_j \right\|_{\infty}$$

We have

$$\left\| g - \sum_{j=1}^n \alpha_j \psi_j \right\|_{\infty} = \text{Max} \left| g(x) - \sum \alpha_j \psi_j(x_j) \right|$$

There fore

$$\|\delta_n^*\| = \underset{\alpha_1 \dots \alpha_n, 0 \leq i \leq m}{\text{MinMax}} \left| g(x_j) - \sum_{j=1}^n \alpha_j \psi_j(x_j) \right| \dots \dots \dots (3-4)$$

Equation (3-4) is treated by solving the following liner-programming problem

Minimize:

$$\left. \begin{aligned} \delta_n(x_i) &= g(x_i) - \sum_{j=1}^n \alpha_j \psi_j(x_i) \\ \text{subject to} \\ \sum_{j=1}^n \alpha_j \psi_j(x_i) - \delta &\leq g(x_i) - \\ \sum_{j=1}^n \alpha_j \psi_j(x_i) - \delta_n &\leq -g(x_i) \\ i &= 1, 2, \dots, m \end{aligned} \right\} \dots \dots \dots (3-5)$$

Where:

$$\psi_j(x_i) = \phi_j(x_i) - \int_0^x k(x_i - y)\phi_j(y)dy.....(3-6)$$

In a liner programming, code is available in which the available are not required to be non – negative, then it can be use on problem as formulated in (3-5).

Moreover the scheme (3-5) solves only problems, which have increased or decreased solution on a fixed interval.

For more general case, in which the solution may oscillate often between positive and negative values, the following modified technique can be applied first, introduces available β_{n+1} and $\alpha_j = \beta_j - \beta_{n+1}$, are written. Then one defines

$$\psi_{n+1}(x_1) = \sum_{j=1}^n \psi_j(x_i)....for.....each...i = 1,.....,n$$

Now the LP problem (3-5) becomes Minimize δ

Subject to :

$$\left. \begin{aligned} \sum_{j=1}^{n+1} \beta_j \psi_j(x_i) - \delta &\leq g(x_i) \\ -\sum_{j=1}^{n+1} \beta_j \psi_j(x_i) - \delta &\leq -g(x_i) \\ \delta \geq 0 \dots \text{and} \dots \beta &\geq 0 \end{aligned} \right\} \dots \dots \dots (3-7)$$

$i=1,2,\dots,m$

The technique for solving VLEs using LP method can be summarized the following steps:

Steps (one)

Use equation (3-2) to consider the approximate solution.

Steps (two)

Compute $\psi_j s(j = 1, \dots, n)$ numerically using Quadrature rules (Trapezoidal and simpson 1/3) for equation (3-6)

Steps(three)

For each $l=1, \dots, m$ use step (2) to compute

$$\psi_{n+1}(x_i) = -\sum_{j=1}^n \psi_j(x_i)$$

Steps(four)

Construct the linear programming problem (3-7)

Steps(five)

To evaluate β_j 's take duality to (3-7) then solve by simplex method.

Steps (six)

Option α_j 's by subtracting β_{n+1} form each

$$\beta_j (j = 1, \dots, n)$$

Steps(seven)

Put the optimal values of α_j 's in equation (3-2) so we have the approximate solution to linear VIEs .

(3-7) We can take example:

example (3-1) consider the problem

$$f(x) = 1 - \int_0^x (x - y) f(y)$$

$f(x) = \cos x$ we seek he approximate solution :

$$f_6(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^4 + \alpha_5 x^4 + \alpha_6 x^5$$

Table (3-1) illustrates the comparison between the exact and numerical solution depending on the least error and running time:

| Linear programming | | | |
|--------------------|----------|---|---|
| x | Exact | Using trapezoidal $F_6(x)$ $=0.999862+0.0009x$ $-0.512x^2+0.0425x^3$ $-0.186x^4+0.0053x^5$ | Using simpson 1/3 $F_6(x) =0.1+0.0008x-$ $-0.5072x^2+0.0212x^3$ $-0.0178x^4+0.0078x^5$ |
| 0 | 1.0 | 1.015380 | 1.0 |
| 0.1 | 0.995004 | 0.996396 | 0.995031 |
| 0.2 | 0.980067 | 0.974699 | 0.980073 |
| 0.3 | 0.955336 | 0.948047 | 0.955328 |
| 0.4 | 0.921061 | 0.914867 | 0.921060 |
| 0.5 | 0.877583 | 0.879112 | 0.877606 |
| 0.6 | 0.825336 | 0.825187 | 0.825381 |
| 0.7 | 0.764824 | 0.767888 | 0.764888 |
| 0.8 | 0.696707 | 0.702342 | 0.696733 |
| 0.9 | 0.621610 | 0.628956 | 0.621627 |
| 1 | 0.540302 | 0.54865 | 0.540400 |
| L.S.E | | 0.000005 | 0.00000 |
| L.S.Eg | | 0.0000095 | 0.00000 |
| R.T | | 0:0:0:5 | 0:0:0:10 |

Table (3-1)

Example (3-2) consider the problem

$$f(x) = 1 + \int_0^x e^{-(x-y)} f(y) \dots \text{which..has..exact..solution}$$

$$f(x) = 1 + x,$$

Put the approximate solution the problem is solved to option the results of approximate function.

Table (3-2) the numerical results with the exact solution as will as least square error and running time

| Linear progrzoidal | | | |
|--------------------|-------|--|---|
| X | Exact | Using Trapezoidal $f_2(x)=1.001293+$ $1.001293x$ | Using simpson 1/3 $f_n(x)=1.0+0.0008x$ |
| 0 | 1.0 | 0.999769 | 0.999990 |
| 0.1 | 1.1 | 1.099898 | 1.099996 |
| 0.2 | 1.2 | 1.200027 | 1.200001 |
| 0.3 | 1.3 | 1.300156 | 1.300006 |
| 0.4 | 1.4 | 1.400286 | 1.400012 |
| 0.5 | 1.5 | 1.588415 | 1.500017 |
| 0.6 | 1.6 | 1.60544 | 1.600022 |
| 0.7 | 1.7 | 1.700673 | 1.700028 |
| 0.8 | 1.8 | 1.800803 | 1.800033 |
| 0.9 | 1.9 | 1.90932 | 1.900038 |
| 1.0 | 2.0 | 2.001061 | 2.000044 |
| L.S.E | | 0.000004 | 0.000000 |
| L.S.Eg | | 0.00000 | 0.000000 |
| R.T | | 0:0:0:10 | 0:0:0:10 |

Table (3-2)

Section 4

Discussion

In this section the linear programming method for solving Integral equation is discussed. The linear programming method provides a very convenient and useful algorithm and the results obtained using this method conform point have been identified:

- 1- Linear programming method can be used even if there is on information about the exact solution (form the residual function S_n in equation(3-3))
- 2- The number of convenient (m) i.e. as m increases the value of the L.S.E approach to zero.
- 3- The order of approximating polynomial (n)
- 4- The involved integrals $\psi_j(x_i)$.

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الخلاصة :

تطوير بعض الطرق التقريبية لحل معادلات فولتيرا التكاملية من الصنف الالتفافي باستخدام البرمجة الخطية حيث تعتمد هذه الطريقة على تقريب الحل باستخدام تقنية جديدة تعتمد على طريقة البرمجة الخطية

(Simplex method) لإيجاد الحل الأفضل لهذه المعادلات عدديا .