Indirect Algorithm for Solving Variation Problems

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Abstract:

In this paper, an approximate indirect method to solve some variational problems is proposed in terms of shifted Legendre polynomials. The operational matrix of differentiation for shifted Legendre polynomials is first derived. Using the operational matrix of differentiation, the variational problems are reduced to the solution of system of algebric equations with unknown shifted Legendre coefficients. Numerical example illustrates the efficiency, simplicity and accuracy of the proposed method.

1. Introduction

Functional minimization problems known as variational problems appear in engineering and science whose minimization of functional, such as Lagrangian, potential, total energy, etc., give the laws govering the systems behavior. In optimal control theory, minimization of certain functionals gives control functions for optimum performance of the system.

Variational problems provide an alternative to search for analytical solutions of some problems. The idea is to formulate variational functional whose stationarity conditions lead to equations that describe the problems. The Euler-Lagrange equations obtained by applying the well known procedure in the calculus of variation, usually leads to equations that are difficult to solve.

Orthogonal functions are special functions in the space of which approximate solutions of variational problems are sought. Several orthogonal functions and wavelets were used to solve variational problems [3,7,8].

In this paper, we shove variational problems using indirect algorithm with the aid of shifted Legendre polynomials. First, some properties of shifted Legendre polynomials are given and then the operational matrix of differentiation is derived and indirect method for solving variational problems is presented.

2. Properties of Legendre Polynomails

Shifted Legendre polynomials are important in approximation theory and numerical analysis and in some quadrature rules that appears in the theory of numerical integration.

Consider the well-known shifted Legendre polynomials of order n ,$\bar{L}_n(t)$,which are orthogonal with respect to the weight function $w(t)=1$ and derived from the following recursive formula

$$\bar{L}_{n+1}(t)=\frac{2n+1}{n+1}(2t-1)\bar{L}_n(t)-\frac{n}{n+1}\bar{L}_{n-1}(t) \qquad n\geq 1$$

where $\bar{L}_0(t)=1,\ \bar{L}_1(t)=2t-1$

2.1 Shifted Legendre Operational Matrix for Differentiation

Considering that the polynomial basis is used to describe a function to be composed of shifted Legendre polynomials, in the interval $[a,b]=[0,1]$, one can observe that,for shifted Legendre polynomials [1].

$$\dot{\bar{L}}_N(x)=\begin{cases}\sum_{r=0}^{(N/2)-1}(8r+6)\bar{L}_{2r+1}(x) & n\ even.0 \\ & N=1,2,3,... \\ \sum_{r=0}^{(N/2)}(8r+2)\bar{L}_{2r}(x) & n\ odd\end{cases} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Difining

$$\bar{L}=\begin{bmatrix}\bar{L}_1\\ \bar{L}_2\\ \vdots\\ \bar{L}_N\end{bmatrix},\ \bar{\overset{*}{L}}=\begin{bmatrix}\bar{L}_o\\ \bar{L}_1\\ \vdots\\ \bar{L}_{N-1}\end{bmatrix},\ \dot{\bar{L}}_o=0$$

One can write $\frac{d}{dx}\bar{L}=D^1\bar{\overset{*}{L}}$

Following (1), D_1 is a square matrix $D^1_{(N-1(\times(N-1)}$ and

$$D^1_{(2i+j),J}=2(2j-1)\ ,\ j=1,2,...,i+1,i=0,1,2,...n \ \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

According to (2) the operational matrix $D^1_{7\times7}$ can bewritten as:

$$D^1_{7\times7}=\begin{pmatrix}2&0&0&0&0&0&0\\0&6&0&0&0&0&0\\2&0&10&0&0&0&0\\0&6&0&14&0&0&0\\2&0&10&0&18&0&0\\0&6&0&14&0&22&0\\2&0&10&0&18&0&26\end{pmatrix}$$

Similarly, one can derive the operational matrix D^2 as follows: Defining $\bar{L}=\left[\bar{L}_2\ \bar{L}_3,...,L_N\ \right]^T, \overset{**}{\bar{L}}=\left[L_o\ L_1,...,L_{N-2}\right]$

$$\frac{d^2}{dx^2}\bar{L} = D^2 \ddot{\bar{L}}, \ddot{\bar{L}}_o = 0, \ddot{\bar{L}}_1 = 0$$

where the operational matrix $D^2_{8\times 8}$ can be obtained as follows:

$$D^2_{8\times 8} = \begin{pmatrix} 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 60 & 0 & 0 & 0 & 0 & 0 & 0 \\ 40 & 0 & 140 & 0 & 0 & 0 & 0 & 0 \\ 0 & 168 & 0 & 252 & 0 & 0 & 0 & 0 \\ 84 & 0 & 360 & 0 & 396 & 0 & 0 & 0 \\ 0 & 324 & 0 & 616 & 0 & 572 & 0 & 0 \\ 144 & 0 & 660 & 0 & 936 & 0 & 780 & 0 \\ 0 & 528 & 0 & 1092 & 0 & 1320 & 0 & 1020 \end{pmatrix}$$

In general, the elements of D^2 can be obtained with the use of the following :

$$D^2_{(2i+j),i} = 4(2j-1)(i+1)(2(i+j)+1)$$

$$j = 1,2,3,\ldots,i \quad , \quad i = 0,1,2,\ldots,N$$

2.2 Function Approximatrion

A function $f(x) \in L^2[0,1]$ may be expanded by the shifted Legendre polynomials series as follows:

$$f(x) = \sum_{r=0}^{\infty} a_r \bar{L}_r(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

where a_r are given by

$$a_r = \frac{< f, \bar{L}_r >}{< \bar{L}_r, \bar{L}_r >}, \qquad r = 0,1,2,\ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

In (4), $<.,.>$ denotes the inner product.

If the infinite series (3) is trancated up to term N, then it can be written as:

$$y(x) \cong \sum_{r=0}^{N} a_r \bar{L}_r(x) = A^T \bar{L}(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

where A and $\bar{L}$ are $(N+1)\times 1$ vectors given by:

$A = \left[a_o\ a_1\ a_2\ \ldots a_N\right]^T$ *and* $\bar{L}(x) = \left[\bar{L}_o\ \bar{L}_1\ \ldots \bar{L}_N\right]^T$, furthermore, x^m can be defined as:

$$1=[1\ 0\ 0\ 0 \quad 0\ 0]\bar{L}(x)=1^{T} L,$$

$$x=\left[\frac{1}{2}\ \frac{1}{2}\ 0\ 0\ \ 0\ 0\right]\bar{L}(x)=x^{T} L,$$

$$x^{2}=\left[\frac{1}{3}\ \frac{1}{2}\ \frac{1}{6}\ 0\ \ 0\ 0\right]\bar{L}(x)=x^{2T} L,$$

$$x^{3}=\left[\frac{1}{4}\ \frac{9}{20}\ \frac{1}{4}\ \frac{1}{20}\ \ 0\ 0\right]\bar{L}(x)=x^{3\,T} L,$$

$$x^{4}=\left[\frac{1}{5}\ \frac{28}{70}\ \frac{20}{70}\ \frac{7}{70}\ \frac{1}{70}\ \ 0\ 0\right]\bar{L}(x)=x^{4\,T} L$$

Finally, differential of the function $y(x)$ defined in (5) can be obtained as:

$$\frac{d}{dx}y(x)=\frac{d}{dx}\sum_{r=0}^{N}a_r\bar{L}_r(x)=a^{T} D^{1}\bar{L}(x)$$

$$\frac{d^{2}}{dx^{2}}y(x)=\frac{d}{dx}\sum_{r=0}^{N}a_r\dot{\bar{L}}_r(x)=a^{T} D^{2}\bar{L}(x)$$

Furthermore, $\frac{d^{n}}{dx^{n}}y(x)=\frac{d^{m-1}}{dx^{m-1}}\sum_{r=0}^{N}a_r\dot{L}_r(x)=a^{T} D^{m}\bar{L}(x)$

3. The Shifted Legendre Indirect Method

Consider the problem of finding the extremum of the functional $x(t)$ The necessary condition for $x(t)$ to extremize is that it should satisfy the Euler-Lagrange equation

$$\frac{\partial f}{\partial x}-\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{x}}\right)=0 \quad \ldots\ldots(6)$$

with appropriate boundary conditions. However, the above differential equation can be integrated easily only for simple cases. Thus numerical and approximate methods have been developed to solve variational problems [2,4,5,6].

In this paper shifted Legendre polynomials are used to establish the indirect method for variational problems.

Suppose, the variable $x(t)$ can be expressed approximately as

$$x(t)=\sum_{i=0}^{m}c_i\bar{L}_i=c^{T}\bar{L}(t) \quad \ldots\ldots(7)$$

Differentiating equation (7) and using (1), we represent $\dot{x}(t)$ as

$$\dot{x}(t)=\sum_{i=0}^{m}c_i\dot{\bar{L}}_i=c^{T}D^{1}\bar{L}(t) \quad \ldots\ldots(8)$$

$$\ddot{x}(t)=\sum_{i=0}^{m}c_i\ddot{\bar{L}}_i=c^{T}D_3^{2}\bar{L}(t) \quad \ldots\ldots(9)$$

We can also express the functions 1,t, in terms of $\bar{L}(t)$ as follows

$$t\cong d^{T}\bar{L}(t) \text{ where } d^{T}=[d_0,d_1,\ldots,d_m] \quad \ldots\ldots(10)$$

The other terms in the functional of equation (6) are known functions of the independent variable t and can be expanded into shifted Legendre polynomial via equation (10)

$u''(x)+bu'(x)=0 \quad u:[0,1],$

For our purpose, we consider the following boundary value problem (6) in the following form

$$-x''(t)+x'(t)=0 \qquad\qquad x:[0,1]\to[0,1] \quad \ldots\ldots\ldots\ldots\ldots\ldots(11)$$

with $x(a)=x_o$ and $x(b)=x_1$

If $\dot{x}(t)=\sum_{i=0}^{n} c_i \bar{L}_i$ is the series that approximates $x(t)$ and D^1, D^2 the operational differentiation matrices, one can write (11) as follows $(-D^2c^T+D^1c^T)^T.\bar{L}(t)=0$

Defining

$$p(t)=\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} \text{ with } c=[c_o c_1 \ldots c_n]$$

By equations the coefficients of $L(t)$ of this matrix equation upon $n-2$ equations to generate a linear algebric equations system with $n-2$ equations and n unknown variables. The two missing equations are obtained from the boundary conditions $x(a)=x_o$ and $x(b)=x_1$.

4. Numerical Results:

In this section, we demonstrate the feasibility and efficiency of the proposed method through several examples.

Example (1): (Harmonic Oscillator)

Consider the following harmonic oscillator

$$\min v[y]=\int_0^1 (y'^2-y^2)dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

Subject to the boundary conditions

$$y(0)=0\,,\ y(1)=1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13)$$

The Euler-Lagrange equation of this problem can be written in the following form $y''=-y$ ……………………………………………….(14)

Whose solution subject to boundary conditions (13) is $y(x)=\dfrac{\sin(x)}{\sin 1}$ To solve eq.(14) by the proposed method, we assume $y(x)$ can be expanded in terms of the shifted Legendre polynomial of fourth order

$$y(x)=\sum_{r=0}^{4} a_r \bar{L}_r(x)=A^T\bar{L}(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15)$$

where $A=[a_o\,a_1\,a_2\,a_3\,a_4]$, $\bar{L}(x)=[L_o\,L_1\,L_2\,L_3\,L_4]^T$. Differentiate eq. (15) twice to obtain $y'(x)$ and $y''(x)$ as follows:

$$y'(x)=\sum_{r=0}^{4}a_r\dot{\bar{L}}_r(x)=A^T D_1\bar{L}(x) \quad \text{……………………………………..(16) and}$$

$$y''(x)=\sum_{r=0}^{4}a_r\ddot{\bar{L}}_r(x)=A^T D_2\bar{L}(x) \quad \text{…………………..…………………(17) where}$$

$$D^1=\begin{pmatrix}0&0&0&0&0\\2&0&0&0&0\\0&6&0&0&0\\2&0&10&0&0\\0&6&0&14&0\end{pmatrix} \text{ and } D^2=\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\12&0&0&0&0\\2&60&10&0&0\\40&6&140&0&0\end{pmatrix}$$

Substituting (15) and (17) into (14) to get $A^T D^2\,\bar{L}(x)=-A^T\bar{L}(x)$ or

$$(A^T D^2+A^T)\bar{L}(x)=0 \quad \text{………………………………………………(18)}$$

with $A=[a_o\,a_1\,a_2\,a_3\,a_4]$, $\bar{L}(x)=[\bar{L}_o\,\bar{L}_1\,\bar{L}_2\,\bar{L}_3\,\bar{L}_4]^T$ the matrix equation (18) is applied to $n-2$ generating a linear algebric equation system with $n-2$ equation and variables. The two missing equations are obtained from the boundary conditions:

$$\left.\begin{array}{l} y(0)=A^T\bar{L}(0)=0\\ y(1)=A^T\bar{L}(1)=1\end{array}\right\} \quad \text{…………………………………………………(19)}$$

Therefore, the algebric system of 5×5 equations is obtained to be:

$$12a_0+40a_4+a_0=0$$

$$60a_3+a_1=0$$

$$140a_4+a_2=0$$

$$a_0-a_1+a_2-a_3+a_4=0$$

$$a_0+a_1+a_2+a_3+a_4=1$$

The solution of this system is

$$a_0=0.0108$$

$$a_1=0.5085$$

$$a_2=0.4927$$

$$a_3=-0.0085$$

$$a_4=-0.0035$$

By substituting the obtained coefficients in (15), the solution of (14) becomes

$$y(x) = 0.0108\bar{L}_0(x) + 0.5085\bar{L}_{\partial 1}(x) + 0.4927\bar{L}_2(x) - 0.0085\bar{L}_3 - 0.0035\bar{L}_4(x)$$

Example (2):

Let us now consider [5], $J(x) = \int_0^1 \left[\dot{x}^2(t) + t\,\dot{x}(t)\right]dt$ ………………….(20)

The boundary conditions are the initial and final conditions

$$x(0) = 0\ ,\ \ x(1) = \frac{1}{4} \quad \text{………………………………………………….(21)}$$

the corresponding Euler-Lagrange equation is

$2\ddot{x} + 1 = 0$ ……………………………..………………………………(22)

Let $x(t) = \sum_{r=0}^{2} a_r \bar{L}_r(t) = A^T \bar{L}(t)$ ………………………………………….(23)

and then

$\dot{x}(xt) = \sum_{r=0}^{2} a_r \dot{\bar{L}}_r(x) = A^T D_1 \bar{L}(x)$ ………………………………………….(24)

$\ddot{x}(t) = \sum_{r=0}^{2} a_r \ddot{\bar{L}}_r(x) = A^T D_2 \bar{L}(x)$ ……………………………………….(25)

where

$A = [a_o\, a_1 a_2]$, $\bar{L}(x) = [L_o\, L_1 L_2]^T$ and the matrices D_1 and D_2 are

$$D^1 = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\ ,\quad D^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 12 & 0 & 0 \end{pmatrix}$$

Next, we substitute (25) into (22) and then simplify to obtain

$24a_2 = -1$ ……………………………………………………………..(26)

The boundary conditions (10) are substituted in (12), yield

$a_0 - a_1 + a_2 = 0$ ………………………………………………..………..(27)

$a_0 + a_1 + a_2 = 1$………………………………………………...…………(28)

The three simultaneous linear equations (26),(28) are used to determine

$$a_0, a_1 \text{ and } a_2\ ,\ a_0 = \frac{1}{6}, a_1 = \frac{1}{8}, a_2 = \frac{-1}{24}$$

If the Euler equation (22) is solved, the exact answer is obtained as

$\dot{x}(t) = \frac{1}{2}(1-t)$ ………………………………………………………...…(29)

$x(t) = \frac{t}{2}(1-\frac{t}{2})$ ……………………………………………………………(30)

The approximate solution is

$x(t) = \frac{1}{6}\bar{L}_0(t) + \frac{1}{8}\bar{L}_1(t) - \frac{1}{24}\bar{L}_2(t)$ ……………………………………..….(31)

Let $E_2 = \|x(t) - x'(t)\|_2 = \left[\int_0^1 |x(t) - \bar{x}(t)|^2 dt\right]^{\frac{1}{2}}$ where $x(t)$ is the exact solution (19) and $\bar{x}(t)$ is the approximate solution (20), and $\|\cdot\|_2$ is the L_2 norm. then $E_2 = 0$, whereas, when Bernestian polynomials is applied, we have $E_2 = 2.19\times10^{-7}$ and the result of [2] when triangular function method is applied is $E_2^8 = 7.132\times10^{-4}, E_2^{256} = 7\times10^{-7}$, $E_2^{512} = 1.0\times10^{-7}$ where E_2^m will denoted the error in $L^2 - norm$ when approximation is performed by using $m \times m$ matrix.

5. Conclusion

The operational matrices of shifted Legendre polynomials for differentiation D^1, D^2 are used to solve variational problems an explicit formula to find the elements of D^1 and D^2 are derived. The present indirect method reduces a variational problems into a set of algebric equation and give satisfactory results comparing with other outhers.

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خوارزمية غير مباشرة لحل مسائل التغاير

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الخلاصة

في هذا البحث، تم اقتراح طريقة تقريبية غير مباشرة لحل بعض مسائل التغاير بدلالة متعددات حدود ليجندر المزاحة. تم اولاً اشتقاق مصفوفة المشتقات لمتعددات حدود ليجندر المزاحة.

وباستخدام هذه المصفوفة يتم اختزال مسائل التغاير الى حل منظومة معادلات جبرية مع عوامل ليجندر المزاحة غير المعلومة، توضح الامثلة العددية كفاءة وسهولة ودقة الطريقة المقترحة.