Least square method for Solving Linear Fredholm and Volterra Integro – Differential Equations of the Second Kind Using Bernstein Polynomial

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Abstract

The main purpose of this paper lies briefly in submitting least square method for solving linear Fredholm and Volterra integro differential equations of the second kind with the aid of Bernstein polynomials as basic functions to compute the approximated solutions of Fredholm and Volterra integro differential equations .Two examples are given for determining the accuracy of the proposed results method.

Keywords: Fredholm and Volterra integro differential equation of the second kind, least square method , Bernstein polynomials, Least square error (L.S.E.).

Introduction

The integro- differential equations represent a rise in a great many branches of science; for example, in potential theory, acoustics, elasticity, fluid mechanics, radioactive transfer, theory of population. Moreover a great many physical problems (e.g., in radiography, spectroscopy, stereo logy, chemical analysis) are appropriately formulated in terms of integro- differential equations.

In many instances the integro-differential equation originates from the conversion of boundary- value problem or an initial- value problem associated with a partial or an ordinary differential equation, but many problems lead directly to integro- differential equations and cannot be formulated in terms of differential equation.

An integro-differential equation is an equation involving one (or more) unknown functions u(x), together with both differential and integral operations on u.

A linear integro-differential equation is an equation of the form

$$u'(x) = f(x) + \lambda \int_a^{b(x)} k(x,t)u(t)dt \qquad \text{…..(1)}$$

Here the kernel k(x,t),f(x) are known functions, u(x) is the unknown function, and λ is a scalar parameter of the equation.
If the limit b(x) in equation (1) is constant (b(x)=b) then equation (1) become

$$u'(x) = f(x) + \lambda \int_a^b k(x,t)u(t)dt \qquad a \le x \le b \qquad \dots\dots.(2)$$

Equation (2) is called **linear Fredholm Integro-differential equation of second kind**, where u(x) is the unknown function ,u'(x) derivatives of u(x) .
The equation (1) is called **linear Volterra Integro-differential equation of second kind**, if b(x)=x which has the form :

$$u'(x) = f(x) + \lambda \int_a^x k(x,t)u(t)dt \qquad \mathbf{a \le x} \qquad \dots\dots(\square)$$

Notation:

For simplicity we can without loss of generality, take $\lambda = 1$
with given initial condition $u(0)$

Bernstein polynomials

Bernstein polynomials are incredibly useful mathematical tools as they are simply defined, can be calculated quickly on computer systems and represent a tremendous variety of functions.
The Bernstein polynomials of degree n are defined by

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i} \qquad \text{for } i=0,\dots,n \qquad , n \ge 0$$

Where

$$\binom{n}{i} = \frac{n!}{i!\,(n-i)!}$$

, n is the degree of polynomials, (i) is the index of polynomials and (t) is the variable. The exponents on the (t) term increase by one as i increases, and the exponents on the (1-t) term decrease by one as (i) increases.
For the numerical solution , the discrete form of the exact solution U(x) of equation (1) can be written as

$$U(x) \cong U_n(x)$$

Where n is positive integer . As we use Bernstein polynomial as a basic function in this paper , So U_n can be written in this form

$$U_n(x) = \sum_{i=0}^{n} c_i B_i^n(x) \qquad \dots\dots(4)$$

Now, by using the operator form , equation (4) can be written as

$$L[U(x)] = f(x)$$

Using the operator form , equation (1)can be written as

$$L[U(x)] = f(x) \qquad \dots\dots(5)$$

Where the operator L is defined for each kind of integro differential equation in equations (2) and (3).
Equations (2) and (3) can be written respectively as below

$$L[u(x)] = u'(x) - \lambda \int_a^b k(x,t)u(t)dt \qquad a \le x \le b$$

$$L[u(x)] = u'(x) - \lambda \int_a^x k(x,t)u(t)dt \qquad \mathbf{a \le x}$$

The unknown function U(x) is approximate by the form

$$U_n(x) \cong \sum_{i=0}^{n} c_i B_i^n(x) \qquad \dots.(6)$$

Now, Substituting equation (6) in (5) we get

$$L[U_n] = f(x) + E_n(x)$$

Where

$$L[U_n(x)] = \sum_{i=0}^{n} c_i \left[u'(x) - \lambda \int_a^{b(x)} k(x,t)u(t)dt\right]$$

For which we have residue equation

$$L[U_n(x)] - f(x) = E_n(x)$$

This equation yield

$$L\left(\sum_{i=0}^{n} c_i B_i^n(x)\right) - f(x) = E_n(x)$$

$$\sum_{i=0}^{N} c_i L\left(B_i^n(x)\right) - f(x) = E_N(x) \dots\dots(7)$$

Obviously the weighting function setting its weight integral equal to zero

Substitute equation (7) in (8)

$$\int w_j \left[\sum_{i=0}^{n} [c_i L\left(B_i^n(x)\right)] - f(x)\right] dx = 0$$

$$\sum_{i=0}^{n} c_i \int w_j L\left(B_i^n(x)\right) dx = \int w_j f(x) dx$$

Where

$$L\left(B_i^n(x)\right) = \frac{dB_i^n(x)}{dx} - \int_a^{b(x)} k(x,t)B_i(t)dt$$

Least square Method

Least square method is one of the approximated methods used to solve Fredholm and Volterra integro differential equations of the second kind .

In this method the weighting function is chosen as follow

$$w_j = L\left(B_j^n(x)\right) \qquad ,j = 1, \ldots\ldots, n$$

This leads to

$$\int L\left(B_j^n(x)\right) L\left(B_i^n(x)\right) c_i = L\left(B_i^n(x)\right) f(x_i) \quad i,j = 1, \ldots, n \qquad \ldots..(9)$$

Hence the above equation can be seen as a system of (N+1) equations in the (N+1) unknown c_i, i=0,1,…,N

….(10)

Hence by using matrices from KA=H

Where

$$K=\begin{bmatrix} \int L(B_0^n)L(B_0^n) & \dots & \int L(B_0^n)L(B_n^n) \\ \vdots & \dots & \vdots \\ \int L(B_n^n)L(B_0^n) & \dots & \int L(B_n^n)L(B_n^n) \end{bmatrix}$$

$$A=\begin{bmatrix} c_0 \\ \vdots \\ c_n \end{bmatrix}$$

And

$$H = \begin{bmatrix} \int L(B_0^n)f(x)dx \\ \vdots \\ \int L(B_n^n)f(x)dx \end{bmatrix}$$

A computationally efficient way to calculate the values c_i's is by solving the system

KA=H
For the coefficient c_i's which satisfies the equation

$$U_n(x) \cong \sum_{i=0}^{n} c_i B_i^n(x) \qquad ....(11)$$

the approximated solution of (1) will be given in the following algorithm
Step 1:

Select
Step 2:

$$L\left(B_i^n(x)\right) = \frac{dB_i^n(x)}{dx} - \int_a^{b(x)} k(x,t) B_i^n(t)dt \qquad i=0,....,n \qquad .....(12)$$

Step 3:
Compute the matrices K , H and solve the system using (10) for the coefficients c_i's
Step 4:
Substitute c_i's in (11) with $B_i^n(x)$ to get the approximate solution $U_n(x)$
Numerical Examples:
1-Consider Volterra Integro-Differential equation of second kind

$$u'(x) = f(x) - \int_a^x u(t)dt \qquad 0 \le x \le 1$$

where f(x)=1 and k(x,t)=1, the exact solution is $u(x) = \sin x$
with initial condition u(0)=0.
We solve the above equation by using the fourth steps with n=2 we get three equation with three c_i's, by using the initial condition and solving the three equations we get
$c_0 = 0$, $c_1 = 0.4852$, $c_2 = 0.9508$
and the approximate solution is
$u_2(x) \cong 0.9704x - 0.0196x^2$
For h=0.1and x=x1=a+ih, i=0,1,...,10
The tabulated result is obtained by applying the method involved in this paper i,e the implementation of Bernstein polynomial; these numerical results are compared with the exact one in the same table (1)
2-Consider Fredholm Integro-Differential equation of second kind

$$u'(x) = f(x) + \int_a^b k(x,t)u(t)dt \qquad 0 \le x \le 1$$

Where $f(x) = xe^x + e^x - x$, k(x,t)=x and a=0,b=1
The exact solution is $u(x) = xe^x$,with initial condition u(0)=0.
We solve the above equation by using the fourth steps in the algorithm we get three equations with three c_i's ,by using the initial condition and solving the three equations we get
$c_0 = 0$, $c_1 = 0.2818$, $c_2 = 2.7184$
and the approximate solution is
$u_2(x) \cong 0.5636x + 2.1548x^2$
the step size h=0.1. The application of Bernstein polynomial yields the results shown in the table below together with the exact solution at each point of *x*.

Conclusion:

This paper present solving linear Fredholm and Volterra integro – differential equations by using Bernstein polynomial with Least square method . Two examples were submitted to illustrate the given idea with good approximate results were archived .
We conclude that:
1- In general , least square method with the aid of Bernstein polynomials have been applied to find the approximate solution of linear Fredholm and Volterra integro –differential equation and have proved their effectiveness from through finding accurate results.
2-From the table (1) and the figure (1) we can see the good result it close to exact solution, also least square error is too small.
3-From the table (2) and the figure (2)see least square error and we can see the good result it seem equal to exact solution.
Also from the examples we can see the smallest least square error .
4- The good approximation solution depends on the size of n of Bernstein polynomial ,where as n increases ,the error term decrease .

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Table (1)
The result of example (1)

X	exact solution	approximate solution
0	0	0
0.1	0.099833	0.096844
0.2	0.198669	0.193296
0.3	0.29552	0.289356
0.4	0.389418	0.385024
0.5	0.479426	0.4803
0.6	0.564642	0.575184
0.7	0.644218	0.669676
0.8	0.717356	0.763776
0.9	0.783327	0.857484
1	0.841471	0.9508
L.S.E. 0.020462		

Figure (1)
Approximation solution of Volterra integro-differetial equation of example (1)

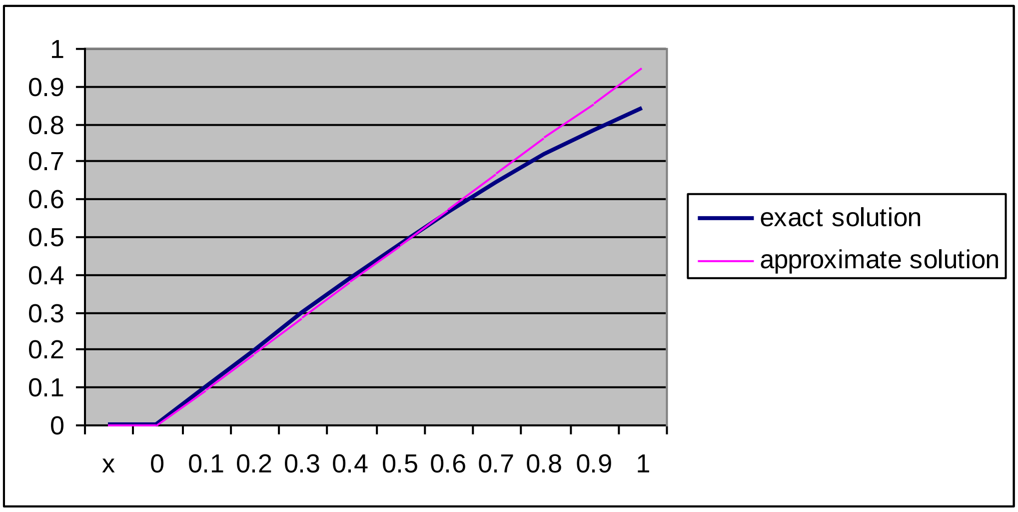
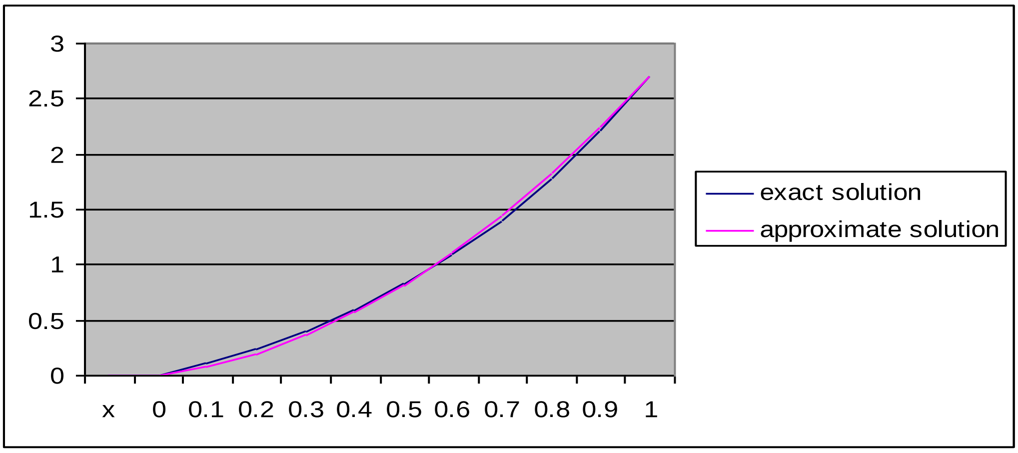


Table (2)
The result of example (2)

x	exact solution	approximate solution
0	0	0
0.1	0.110517	0.077908
0.2	0.244281	0.198912
0.3	0.404958	0.363012
0.4	0.59673	0.570208
0.5	0.824361	0.8205
0.6	1.093271	1.113888
0.7	1.409627	1.450372
0.8	1.780433	1.829952
0.9	2.213643	2.252628
1	2.718282	2.7184
L.S.E.	0.011657	

Figure (2)
Approximation solution of Fredholm integro-differetial equation of example (2)



طريقة المربعات الصغرى لحل معادلات فريدهولم وفولتيرا التكاملية التفاضلية الخطية من النوع الثاني باستخدام متعددة حدود برنشتاين.

الخلاصة

الهدف الرئيسي من بحثنا المقدم هو تقديم طريقة المربعات الصغرى لحل معادلات فريدهوم وفولتيرا التكاملية التفاضلية الخطية ومن النوع الثاني باستخدام متعددة حدود برنشتاين كدالة أساسية في الحل لإيجاد الحل التقريبي باستخدام الطريقة أعلاه. تم اعطاء مثاليين لبيان كيف إن الحل يتقارب بسرعة وخطوات الحل قليلة باستخدام هذه الطريقة المقترحة .