The Quaternions and Representing Methods

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Abstract

In this paper I deal with quaternions history which were discovered by William Rowan Hamilton in 1843.

Also I take some definitions and the basic properties of quaternions.

My basic aim is how to representing quaternions as matrices. There are at least two ways of the representing of matrices, the first way is used 2×2 complex matrices and the second one is used 4×4 real matrices.

I can able to generalize the first way by using n×n complex matrices that n is even.

Introduction

In mathematics, the quaternions are a number system that extends the complex numbers. They were first described by the Irish mathematician Sir William Rowan Hamilton in 1843 and applied to mechanics in three-dimensional spaces.

A feature of quaternions is that the product of two quaternions is non-commutative. Hamilton defined a quaternions of two directed lines in a three-dimensional space [6]. or equivalently as the quotient of two vectors [7]. Quaternions can also be represented as the sum of scalar and vector.

Quaternions find uses in both theoretical and applied mathematics, in particular for calculations involving three-dimensional rotations such as in three-dimensional computer graphics and computer vision. They can be used alongside other methods, such as Euler angles and matrices, or as an alternative to them depending on the application.

In modern Language, quaternions forms four-dimensional associative normed division algebra over the real numbers, and thus also form a domain. In fact, the quaternion were the first non-commutative division algebra to be **H** (for Hamilton).

Definition and some basic Properties:

Definition 1 [2]

A quaternion is a vector z=a+bi+cj+dk with real cofficients, a,b,c,d, the product of any two of the quaternions 1,i,j,k is defined by the requirement that 1 act as an identity and by the table:

$i^2 = j^2 = k^2 = -1$

ij=+ji=k, jk=-kj=j, ki=-ik=j

Now I can use the following multiplication table to represent these relations:

•	1	i	j	k
1	1	i	j	k
i	i	-1	k	-j
j	j	-k	-1	i
k	k	j	-i	-1

[11]

Definition 2[5]

If. z_1,z_2 are two quaternions, then we can say that $z_1=z_2$ if their coordinates are equal.

Addition and multiplication:

Let z_1, z_2 be two quaternions, then,

$$z_1+z_2=(a_1+b_1j+c_1j+d_1k)+(a_2+b_2i+c_2j+d_2k)$$
$$=(a_1+a_2)+(b_1+b_2)i+(c_1+c_2)j+(d_1+d_2)k$$

And $z_1z_2=(a_1+b_1i+c_1j+d_1k)(a_2+b_2i+c_2j+d_2k)$ ……………(*)

$$=a_3+b_3i+c_3j+d_3k$$

Where,

$$\left.\begin{aligned} a_3&=a_1a_2-b_1b_2-c_1c_2-d_1d_2\\ b_3&=a_1b_2+b_1a_2+c_1d_2-d_1c_2\\ c_3&=a_1c_2-b_1d_2+c_1b_2+d_1a_2 \times c_1a_2+s_1b_2\\ d_3&=a_1d_2+b_1c_2-c_1b_2+d_1a_2 \end{aligned}\right\} \quad (**)$$

Definition 3 [1]

The quaternion z=a-bi-cj-dk is called the conjugate of

Z=a+bi+cj+dk

Definition 4

The absolute value of z=a+bi+cj+dk is non negative real number whose defined by

$$|z| = \sqrt{\overline{zz^*}} = \sqrt{a^2+b^2+c^2+d^2}$$

Lemma 1[9]

The conjugate of quaternion z is satisfied the following properties:

1. $(z^*)^*=z$
2. $(\alpha z_1+\beta z_2)^*=az_1^*+bz_2^*$ where ,$\beta\in R$
3. $(z_1z_2)^*=z_2^*z_1^*$

Proof (1)

If z=a+bi+cj+dk

Then z^*=a-bi-cj-dk

Therefor $(z^*)^*$=a+bi+cj+dk

Proof (2)

If $z_1=a_1+b_1i+c_1j+d_1k$

$z_2=a_2+b_2i+c_2j+d_2k$

and if $\alpha,\beta \in R$

then $\alpha z_1=\beta z_2+(\alpha a_1+\beta a_2)+(\alpha b_1+\beta b_2)i+(\alpha c_1+\beta c_2)j+(\alpha d_1+\beta d_2)K$

by Definition (3) then

$(\alpha z_1+\beta z_2)^*=(\alpha a_1+\beta a_2)-(\alpha b_1+\beta b_2)i-(\alpha c_1+\beta c_2)j-(\alpha d_1+\beta d_2)$

$=\alpha(a_1-b_1i-c_1j-d_1k)+\beta(a_2-b_2i-c_2j-d_2k)$

$=\alpha z^*+\beta z^*_2$

Proof (3)

To prove third property, it follows form (*),(**) and def(3), so that

$(z_1z_2)^*=z_2^*z_1^*$

Definition (5)

If z is any quaternion, then the norm of z which is denoted by N(z) is given by

$N(z)=zz^*$ that is $z\neq 0$

Now

$N(z)=(a+bi+cj+dk)(a-bi-cj-dk)$

$=a^2+b^2+c^2+d^2$

So that N(z) is a positive real number and N(0)=0

Lemma (2)[9]

If z_1,z_2 are two quaternious, then,

$N(z_1z_2) = N(z_1)N(z_2)$

Proof: By definition of the norm

$N(z_1z_2)=z_1z_2.(z_1z_2)^*$

And by the third part of lemm(1)

$(z_1z_2)^*=z_2^*z_1^*$

So that $N(z_1z_2)=z_1z_2.(z_1z_2)^*$

Since $N(z_2)=z_2z_2^*$

Where $N(z_2)$ is commutative with z_1^*

Then $N(z_1z_2)=z_1\,(z_2z_2)^*z_1^*$

$= (z_1z_1^*)(z_2z_2^*)$

$=N(z_1)N(z_2)$

This result is lead us to (Lagrange theorem)

Lagrange theorem

If a_1,b_1,c_1,d_1, and a_2,b_2,c_2,d_2 are rational numbers then:

$(a_1^2+b_1^2+c_1^2+d_1^2)(a_2^2+b_2^2+c_2^2+d_2^2)=$

$(a_1a_2-b_1b_2-c_1c_2-d_1d_2)^2+(a_1b_2+b_1a_1+c_1d_2-d_1c_2)^2$

$+(a_1c_2-b_1d_2+c_1a_2+d_1b_2)^2+(a_1d_2+b_1c_2-c_1b_2+d_1a_2)^2$

Proof:

Since the left side is equal $N(z_1)N(z_1)$

and the right side is equal $N(z_1z_2)$

hence by lemma (2)

$N(z_1)N(z_2)=N(z_1z_2)$

Definition (6): [10]

The quaternion that's satisfied the equation:

$x^n+a_1x^{n-1}+a_2x^{n-2}$……..$+a_2=0$ where ai are integer numbers, is called integral quaternion .

Definition (7): [8]

We defined z^{-1} when $z\neq 0$ by

$$z^{-1}=\frac{z^*}{N(z)}$$

so that

$z.z^{-1}=z^{-1}.z=1$

If z and $z^{-1}$are both integral, then we say that z is a unity and write

$Z=\in$, since $zz^{-1}=1$, $n(\in)N(\in)^{-1}=1$

And so $N(\in)=1$.

Then $z^{-1}=z^*$ is also integral, so that z is a unity.

Thus a unity may be defined alternatively as an integral quaternion whose norm m is 1.

Representing quaternions by matrices

There are at least two ways of representing quaternions as matrices, in such a way that quaternion addition and multiplication correspond to matrix addition and matrix multiplications (homomorphisms). One is to use 2*2 complex matrices, and the other is to use 4*4 real matrices.

In the first way, the quaternion a+bi+cj+dk is represented as :

$$\begin{bmatrix} a+d & b+ci \\ -b+ci & a-ci \end{bmatrix}$$

This representation has several nice properties:

- Complex numbers (c=d=0) correspond to diagonal matrices.
- The square of the absolute value of the quaternion is the determinant of corresponding matrix.
- The conjugate of quaternion corresponds to the conjugate transpose of the matrix, i.e. a*.

Lemma (3) [3]

Quaternions can be represented using the complex 2*2 matrices:

Proof :

Let $\alpha=\left\{a\begin{bmatrix}1&0\\0&1\end{bmatrix}+b\begin{bmatrix}0&1\\-1&0\end{bmatrix}+c\begin{bmatrix}0&i\\i&0\end{bmatrix}+d\begin{bmatrix}i&0\\0&-i\end{bmatrix} a,b,c,d\in R\right\}$

Then

$$\begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}0&1\\-1&0\end{bmatrix}=\begin{bmatrix}-1&0\\0&-1\end{bmatrix}=-I$$

$$\begin{bmatrix}0&i\\i&0\end{bmatrix}\begin{bmatrix}0&i\\i&0\end{bmatrix}=\begin{bmatrix}-1&0\\0&-1\end{bmatrix}=-I$$

$$\begin{bmatrix}i&1\\0&-i\end{bmatrix}\begin{bmatrix}i&i\\0&-i\end{bmatrix}=\begin{bmatrix}-1&0\\0&-1\end{bmatrix}=-I$$

Therefore

$$\begin{bmatrix}0&1\\-1&0\end{bmatrix}^2\begin{bmatrix}0&i\\i&0\end{bmatrix}^2=\begin{bmatrix}i&0\\0&-i\end{bmatrix}\quad=-I$$

and also

$$\begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}0&i\\i&0\end{bmatrix}=\begin{bmatrix}i&0\\0&-i\end{bmatrix}$$

$$-\begin{bmatrix}0&i\\i&0\end{bmatrix}\begin{bmatrix}0&1\\-1&0\end{bmatrix}=\begin{bmatrix}i&0\\0&-i\end{bmatrix}$$

so that

$$\begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}0&i\\i&0\end{bmatrix}=-\begin{bmatrix}0&i\\i&o\end{bmatrix}\begin{bmatrix}0&1\\-1&0\end{bmatrix}=\begin{bmatrix}i&0\\0&-i\end{bmatrix}$$

in the same way we see that

$$\begin{bmatrix}0&i\\i&0\end{bmatrix}\begin{bmatrix}i&0\\0&-i\end{bmatrix}=-\begin{bmatrix}i&0\\0&-i\end{bmatrix}\begin{bmatrix}0&i\\i&0\end{bmatrix}=\begin{bmatrix}0&1\\-1&0\end{bmatrix}$$

and

$$\begin{bmatrix}i&0\\0&-i\end{bmatrix}\begin{bmatrix}0&1\\-1&0\end{bmatrix}=-\begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}i&0\\0&-i\end{bmatrix}=\begin{bmatrix}0&i\\i&0\end{bmatrix}$$

therefore α is quaternion.

Now i can generalization this lemma over complex n*n matrices such that n is even integer.

Lemma (4)

Quaternions can be represented using the complex n*n matrices such that n is even.

$$\begin{pmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix} & \begin{bmatrix}0&-1\\1&0\end{bmatrix}\end{pmatrix}$$

$$a\begin{bmatrix} \begin{matrix}1&0\\0&1\end{matrix} & & 0 \\ & \ddots & \\ 0 & & \begin{matrix}1&0\\0&1\end{matrix}\end{bmatrix} + b\begin{bmatrix} \begin{matrix}0&-1\\1&0\end{matrix} & & 0 \\ & \ddots & \\ 0 & & \begin{matrix}0&-1\\1&0\end{matrix}\end{bmatrix}$$

$$+c\begin{bmatrix} \begin{matrix}0&i\\i&0\end{matrix} & & & 0 \\ & \begin{matrix}0&i\\i&0\end{matrix} & & \\ & & \ddots & \\ 0 & & & \begin{matrix}0&i\\i&0\end{matrix}\end{bmatrix} + d\begin{bmatrix} \begin{matrix}i&0\\0&-i\end{matrix} & & & 0 \\ & \begin{matrix}i&0\\0&-i\end{matrix} & & \\ & & \ddots & \\ 0 & & & \begin{matrix}i&0\\0&-i\end{matrix}\end{bmatrix}$$

$a,b,c,d \in R$

<u>Proof:</u>

We can use the same proof of lemma (3).

The second way the quaternion a+bi+cj+dk is represented as:

$$\begin{bmatrix} a & -b & d & -c \\ b & a & -c & -d \\ -d & c & a & -b \\ c & d & b & a \end{bmatrix}$$

In this preposition, the conjugate of quaternion corresponds to the transpose of the matrix. The fourth power of the absolute value of quaternion is determinant of corresponding matrix.

In the other words, in R^4 The basis of quaternion can be given by: [1].

$$i = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ \end{bmatrix} \quad j = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ \end{bmatrix}$$

0 0 1 0 1 0 0 0

$$1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad k = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The quaternion satisfies the following identities, sometimes known as Hamilton's rules.

$I^2 = j^2 = k^2 = -1$

ij = -ji = k

jk = -kj = i

ki = -ik =j

Construction of quaternion from complex numbers

According to Cayel-Dickson construction, a quaternion is an ordered pair of complex numbers. Letting **j** be anew root of -1, different from both **i** and **–i** and giving **u** and **v** are pair of complex numbers, then

q= u +jv

Is a quaternion.

If u = a + ib and v = c + id then

q = a+ ib +jc + jid

moreover let

ji = -ij

so that

q = a+ ib +jc + ij (-d)

and also let the product of quaternion be associative

With these rules, we can now derive the multiplication table i,j and ij, the imaginary components of a quaternion:

ii = -1

ij = (ij),

i(ij) = (ii)j = -j

ji = - (ij),

jj = -1

j(ij) = -j(ji) = -(jj)i = i,

(ij)i = -(ji)i = -j(ii) = j,

(ij) = i(jj) = -i

(ij)(ij) = - (ij)(ji) = -i(jj)i = ii = -1

notice how the dyad **i,j** behaves just like the **k** in the definition.

For any complex number v=c+id , its product with j has the following property:

jv = v*j

Since

jv = jc+jid = jc-(ij)d = (c-id)j = v*j

Let p be the quaternion with complex components w and z

p = w +jz

qp = (u + jv) (w + jz) = uw + ujz + jvw + jvjz

= uw + ju*z + jvw + jjv*z

= (uw – v*z) + j(u*z + vw)

since the product of complex numbers is commutative

we have

(u+jv)(w+jz) = (uw – zv*) + j(u*z + wv)

Which is precisely how quaternion multiplication is defined by the Cayley- Dckson construction.

Note that if u=a+ib , v=v+id, and p=a+ib+jc+kd then construction from u and v is rather

P= u +vs = u+jv*

Generalization

Main article : quaternion algebra

If F is any field with characteristic different from two elements **a** and **b** of F , one may define a four dimensional unitary associative algebra over F by using tow generators i and j and the relations i^2=aj^2=b and ij=-ji. This algebra is called quaternion isomorphic to the algebra 2*2 matrices over F, or they are division algebra over F.

History

Quaternion algebra was introduced by Irish mathematician Sir William Rowan Hamilton in 1843. Important precursors to the work included euler's four-square identity (1748) and Olined Rodrige's parameterzation of general fotations by four parameters (1840), but neither of thesw writers treated the four-prameter rotations as an algebra [4][1] carl friedrich gauss had also discovered queternions in 1819, but this work only published in 1900[12].

Hamilton knew that the complex numbers cauld be inter preted as points in a plane, and he was looking for a way to do the same for points in three-dimensional space.

Points in space can be represented by their coordinates, which are triples of numbers, and for many years Hamilton had known how to add and subtract triples of numbers.

However, Hamilton had been stuck on the problem of multiplication and division for a long time. He cald not figure out how to calculate the quotient of the coordinates of two points in space.

The great vreak through in quaternions finally came on Monday 16 October 1843 in Oublin, when Hamilton was on his way to he Royal Irish Academy where he was going to preside at a council meeting. While along the towpath of the Royal Camal with his wife, the concepts behind quaternions were taking shape in his mind. When the answer dawned on him, Hamilton could not resist the urge to carve the formula for the quaternions

$i^2=j^2=k^2=-1$

into the stone brougham bridge as he paused on it. On following day. Hamilton wrote a letter to his friend and fellow mathematician, John T. Graves, describing the train of thought that led to his discovery. The letter was later published in the London, Edinburgh, and Dublin.

Philosophical Mazine and Journal of Science, Vol.IIV (1844, pp. 489-95). On the letter, Hamilton states, and her there dawned on me the nation that we must admit, in some sense, a fourth dimension of space for the purpose of calculating with triples an electric circuit seemed to close, and spark flashed forth.

Hamilton called a quadruple with these rules of multiplication a quaternion, and he devoted most of the remainder of his life to studying and teaching them.

He founded a school of "quaternionsists" and he tired to popularize quaternion in several books. The last and longest of his books, "elements of Quaternion, was 800 pages long and was published shortly after his death.

After Hamilton's death, his student peter tait continued promoting quaternions. At this time, quaternions were amendatory examination topic in Dublin. Topics in phiscs and geometry that would now ve described using vectors, such as kinematics in space and Maxwell's equations, were describe bed entirely in terms of quaternions. There was even a professional research association, the quaternion society, devoted to the study of quaternions and other hyper complex number system. From the mid-1880s, quaternions began to be displaced by vector analysis, which had been developed by Josian Willard Gibbs, Oliver Haviside, and Helmmann von Helmholtz vector analysis described the same phenomena as quaternions, so it borrowed some ideas and terminology liberally from the literature of quaternions. However, vector analysis was conceptually simpler and not atonally cleaner, and eventually quaternions were relegated to a minor in math emetics and physics. A side-effect of this transition is Hamilton's work is difficult to compreheud for many for many modern readers. Hamiltons original definitions are unfamiliar and his writing style was prolix and opaque.

However, quaternions have rad a revival since the 20^{th} century, primarily due to their utility in describing spatial rotations. The representation of rotation by quaternions are more compact and quicker to compute than the representations by matrices, unlike Euler angles they are not susceptible to gimbal lock. For this reason, quaternions are used in computer graphics coputer vision, robotics, control theory, single processing, attitude control, physics, bioinformatics, molecular dynamics, coputer simulations, and orbital mechanics. For example, it is common for the attitude-control systems of space craft to be commanded in terms of quaternions have received another boost from number theory because of their relation ships with the quadratic forms.

Since 1989, the department of mathematics of the national university of Ireland, Maynooth has organized a pilgrimage, where scientists (including the phyiscists murray Gell-Mann in 2002, Steven Weinberg in 2005 and the mathematicico Anderw in 2003) take a walk from Dunsink Observatory to the Royal Canal bridge where no trace of Hamilton's carving remains, unfortunately.

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الرباعيات وطرق تمثيلها

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الملخص

في هذا البحث تطرقت الى تاريخ الرباعيات حيث انها اكتشفت من قبل العالم الرياضي وليليم روان هاملتون (1893) كما انني تناولت بعض التعاريف والخواص الاساسية التي تتعلق بالرباعيات.

وقد كان هدفي الاساس في هذا البحث هو كيفية تمثيل الرباعيات كمصفوفات فهنالك على الاقل طريقتان: الاولى: استخدمت فيها مصفوفات عقدية سعة 2×2 والثانية استخدمت فيها مصفوفات حقيقية سعة 4×4 وقد استطعت تعميم الطريقة الاولى باستعمال مصفوفات عقدية سعة ن×ن حيث ان ن عدد زوجي.