On bounded operator equation

$$A^* XB^* + BX^* A = C$$

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ABSTRACT

In this paper, we give the general solutions of bounded operator equation $A^* XB^* + BX^* A = C$ (1) , where A and B are noninvertible operator .on complex Hilbert space H and properties of the mapping $\mu_{A,B}(X) = A^* XB^* + BX^* A$

INTRODUCTION

In 2007 D. S. Djordjevic [2] , find the explicit solution of operator equation $A^* X + X^* A = B$ for linear operators on Hilbert spaces. Dragana S. in 2008 generalized the result of D. S. Djordjevic in to operator equation $AXB + B^* X^* A^* = C$ [1] , and study solvability of this operator equation .The purpose of this paper is modify the operator equation appear in [1] and give the necessary and sufficient conditions to get the general explicit solution of bounded operator equation $A^* XB^* + BX^* A = C$ where A and B are noninvertible operator , as well as studied some properties of nonlinear operator mappings $\mu_{A,B}(X) = A^* XB^* + BX^* A$, $X \in B(H)$. Let H be arbitrary complex Hilbert space , $B(H)$ be the space of all bounded linear operators from H into H. Let $\mu_A : B(H) \to B(H)$, be the mapping defined as follow $\mu_{A,B}(X) = A^* XB^* + BX^* A$, $X \in B(H)$,where A and B is known operators in $B(H)$ .but X is unknown operator must be determine and then Rang(μ_A)={ $A^* XB^* + BX^* A, X \in B(H)$ } . Also, here we need recall some basic concept of operator that the adjoint operator A^* of $A \in B(H,K)$ is the operator $A^* : K \to H$ such that $< Ax, y >=< x, A^* y >$, where for all $x \in H$ and $y \in K$, an operator A is said to be self-adjoint if $A^* = A$, and skew-adjoint if $A^* = -A$,[4], The moor-penrose inverse of $A \in B(H,K)$ is defined as the operator $A^+ \in B(K,H)$ satisfying the equations $AA^+ A = A$, $A^+ AA^+ = A^+$, $(AA^+)^* = AA^+$, $(A^+ A)^* A^+ A$, also for the mapping $\mu_{A,B}$ if $\mu_{A,B}(XY) = X \mu_{A,B}(Y) + \mu_{A,B}(X)Y$, for all $X, Y \in B(H)$, then the mapping $\mu_{A,B}$ is derivation .

1- The solution of operator equation $A^* XB^* + BX^* A = C$:

in this section , we give the general solution for the operator equation : $A^* XB^* + BX^* A = C$ **…(1) ,** where A, B, C are known and X is unknown operator on H that must be determined . The following theorem introduce the general solution for the nonlinear operator equation (1) by giving the necessary and sufficient conditions, where A and B are bounded invertible operator define on Hilbert space H .

Theorem (1.1):

Let $A \in B(H, K)$ and $B \in B(K, H)$ be an invertible operators if $C \in B(K)$ is a self-adjoint operator. then $X = \frac{1}{2}(A^*)^{-1} C (B^*)^{-1} + (A^*)^{-1} Z (B^*)^{-1}$ where $Z \in B(H)$ is a skew-adjoint operator.

Proof:

Let X be any solution of equation (1), then $A^* XB^* + BX^* A = C$
and hence, $X = (A^*)^{-1} C (B^*)^{-1} - (A^*)^{-1} BX^* A (B^*)^{-1}$

$$= \frac{1}{2}(A^*)^{-1} C (B^*)^{-1} + \frac{1}{2}(A^*)^{-1} C (B^*)^{-1} - (A^*)^{-1} BX^* A (B^*)^{-1}$$

$$= \frac{1}{2}(A^*)^{-1} C (B^*)^{-1} + \left[\frac{1}{2}(A^*)^{-1} C (B^*)^{-1} - (A^*)^{-1} BX^* A (B^*)^{-1} \right]$$

$$= \frac{1}{2}(A^*)^{-1} C (B^*)^{-1} + (A^*)^{-1} \left[\frac{1}{2} A^* XB^* + \frac{1}{2} BX^* A - BX^* A \right] (B^*)^{-1}$$

$$= \frac{1}{2}(A^*)^{-1} C (B^*)^{-1} + (A^*)^{-1} \left[\frac{1}{2} AXB - \frac{1}{2} BX^* A \right] (B^*)^{-1}$$

Let $Z = \frac{1}{2} A^* XB^* - \frac{1}{2} BX^* A$ then $Z^* = \left[\frac{1}{2} A^* XB^* - \frac{1}{2} BX^* A \right]^*$

$$= \left[\frac{1}{2} BX^* A - \frac{1}{2} A^* XB^* \right]$$

$$= -Z.$$

Therefore; Z is a skew-adjoint operator

Then any solution of operator equation (1) has the form $X = \frac{1}{2}(A^*)^{-1} C (B^*)^{-1} + (A^*)^{-1} Z (B^*)^{-1}$

Now, The following proposition give the sufficient condition shows that converse of above theorem is true

Proposition (1.2):

Let $A \in B(H, K)$ and $B \in B(K, H)$ be invertible operators. If $X = \frac{1}{2}(A^*)^{-1} C (B^*)^{-1} + (A^*)^{-1} Z (B^*)^{-1}$ is a general solution of equation (1) then $C \in B(K)$ is a self -adjoint operator**,** where $Z \in B(H)$ is a skew-adjoint operator.

Proof:

Assume that $X = \frac{1}{2}(A^*)^{-1}C(B^*)^{-1} + (A^*)^{-1}Z(B^*)^{-1}$ is a general solution of operator equation (1) then its satisfy this equation, thus we get:

$$A^*\left(\frac{1}{2}(A^*)^{-1}C(B^*)^{-1} + (A^*)^{-1}Z(B^*)^{-1} \right)B^* + B\left(\frac{1}{2}(A^*)^{-1}C(B^*)^{-1} + (A^*)^{-1}Z(B^*)^{-1} \right)^* A = C$$

and reduces into $C = C^*$, therefore; $C \in B(H)$ is a self -adjoint operator.
Now, from above theorem (1.1) its easy to get the following corollary

Corollary (1.3):

Let $A \in B(H,K)$ and $B \in B(K,H)$ be an invertible operators and $Z \in B(H)$ is skew adjoint operator. Then $X = \frac{1}{2}(A^*)^{-1}C(B^*)^{-1} + (A^*)^{-1}Z(B^*)^{-1}$ is a general solution of equation $A^*XB^* - BX^*A = C$ if and only if $C \in B(K)$ is a skew-adjoint operator.

Now, we give the general solution of nonlinear operator equation (1), when B is invertible operator and A is noninvertible operator

Theorem (1.4):

Let $A \in B(H,K)$ and $B \in B(K,H)$ be an operators and A has closed range. Then equation (1) has solution if and only if $C = C^*$ and $(I - D^+D)E(I - D^+D) = 0$, where $D = A(B^{-1})^*$, $E = B^{-1}C(B^*)^{-1}$.

Proof:

In first we reduced equation (1) in to $B^{-1}A^*X + X^*A(B^*)^{-1} = B^{-1}C(B^{-1})^*$

We claim $X = \frac{1}{2}(D^*)^+E + \frac{1}{2}(D^*)^+E(I - D^+D) + (Z - Z^*)D$ is a general solution of equation (1). To do this substitute in the left side of operator equation $D^*X + X^*D = E$ then we get:

$$\begin{aligned} D^*X + X^*D &= D^*\left(\frac{1}{2}(D^*)^+E + \frac{1}{2}(D^*)^+E(I - D^+D) + (Z - Z^*)D \right) \\ &+ \left(\frac{1}{2}(D^*)^+E + \frac{1}{2}(D^*)^+E(I - D^+D) + (Z - Z^*)D \right)^* D \\ &= \frac{1}{2}(D^+D)^*E + \frac{1}{2}(D^+D)^*E(I - D^+D) + D^*ZD - D^*Z^*D \\ &+ \frac{1}{2}E^*D^+D + \frac{1}{2}(I - (D^+D)^*)E^*D^+D + D^*Z^*D - D^*ZD \end{aligned}$$

And by using the condition $C = C^*$ one can have :

$$= \frac{1}{2}D^+DE + \frac{1}{2}D^+DE(I - D^+D) + \frac{1}{2}ED^+D + \frac{1}{2}(I - D^+D)ED^+D$$

$$= \frac{1}{2} D^+ DE + \frac{1}{2} D^+ DE - \frac{1}{2} D^+ DED^+ D + \frac{1}{2} ED^+ D + \frac{1}{2} ED^+ D - \frac{1}{2} D^+ DED^+ D$$
$$= ED^+ D + D^+ DE - D^+ DED^+ D$$
$$= E$$

Conversely, since $A^* XB^* + BX^* A = C$ so, $\left(A^* XB^* + BX^* A\right)^* = C^*$, therefore; $BX^* A + A^* XB^* = C^*$, then, $C = C^*$.

Also , $(I - D^+ D)E(I - D^+ D) = (I - D^+ D)(D^* X + X^* D)(I - D^+ D)$

$$= [D^* X + X^* D - D^+ DD^* X - D^+ DX^* D] \cdot (I - D^+ D)$$
$$= [D^* X + X^* D - (D^+ D)^* D^* X - D^+ DX^* D] \cdot (I - D^+ D)$$
$$= [D^* X + X^* D - D^* (D^+)^* D^* X - D^+ DX^* D] \cdot (I - D^+ D)$$
$$= [D^* X + X^* D - D^* X - D^+ DX^* D] \cdot (I - D^+ D)$$
$$= [X^* D - D^+ DX^* D] \cdot (I - D^+ D)$$
$$= X^* D - D^+ DX^* D - X^* DD^+ D + D^+ DX^* DD^+ D$$
$$= X^* D - D^+ DX^* D - X^* D + D^+ DX^* D$$
$$= 0$$

Now, we give the general solution of nonlinear operator equation when A is invertible operator and B is noninvertible operator

Theorem (1.5):

Let $A \in B(H, K)$ and $B \in B(K, H)$ be an operators and B has closed range. Then equation (1) has solution if and only if $C = C^*$ and $(I - DD^+)E(I - DD^+) = 0$, where $D = (A^*)^{-1} B$, $E = (A^*)^{-1} CA^{-1}$.

Proof:

In first we reduced equation (1) in to $XB^* A^{-1} + (A^*)^{-1} BX^* = (A^*)^{-1} CA^{-1}$

We claim $X = \frac{1}{2} E(D^+)^* + \frac{1}{2} (I - DD^+)E(D^+)^* + D(Z - Z^*)$ is a solution of equation (1). To do this substitute in the left side of operator equation $XD^* + DX^* = E$ then we get:

$$XD^* + DX^* = \left(\frac{1}{2} E(D^+)^* + \frac{1}{2} (I - DD^+)E(D^+)^* + D(Z - Z^*) \right) D^*$$
$$+ D\left(\frac{1}{2} E(D^+)^* + \frac{1}{2} (I - DD^+)E(D^+)^* + D(Z - Z^*) \right)^*$$
$$= \frac{1}{2} E(D^+)^* D^* + \frac{1}{2} (I - DD^+)E(D^+)^* D^* + DZD^* - DZ^* D^*$$
$$+ \frac{1}{2} DD^+ E^* + \frac{1}{2} DD^+ E^* (I - DD^+)^* + DZ^* D^* - DZD^*$$
$$= \frac{1}{2} E(DD^+)^* + \frac{1}{2} E(DD^+)^* - \frac{1}{2} DD^+ E(DD^+)^* + \frac{1}{2} DD^+ E^* + \frac{1}{2} DD^+ E^* - \frac{1}{2} DD^+ E^* (DD^+)^*$$

And by using the condition $C = C^*$ one can have:

$$= \frac{1}{2} EDD^+ + \frac{1}{2} EDD^+ - \frac{1}{2} DD^+ EDD^+ + \frac{1}{2} DD^+ E + \frac{1}{2} DD^+ E - \frac{1}{2} DD^+ EDD^+$$
$$= EDD^+ + DD^+ E - DD^+ EDD^+$$
$$= E$$

Conversely, since $A^* XB^* + BX^* A = C$ so, $\left(A^* XB^* + BX^* A\right)^* = C^*$**,** therefore; $BX^* A + A^* XB^* = C^*$**,** then, $C = C^*$.

Also, $(I - DD^+)E(I - DD^+) = (I - DD^+)(XD^* + DX^*)(I - DD^+)$
$$= (I - DD^+)(XD^* + DX^* - XD^* - DX^* DD^+)$$
$$= DX^* - DX^* DD^+ - DX^* + DX^* DD^+$$
$$= 0$$

Now, we give the general solution of nonlinear operator equation when A and B are noninvertible operator

Theorem (1.6):

Let $A \in B(H,K)$, $B \in B(K,H)$ and D be an operators have closed range then equation (1) has general solution if and only if $C = C^*, (I - D^+ D)E(I - D^+ D) = 0$, where $D = A(B^+)^*$, $E = B^+ C(B^+)^*$ and $A^* BB^+ = BB^+ A^*$.

Proof:

In first we reduced equation (1) in to $B^+ A^* XB^* (B^+)^* + B^+ BX^* A(B^+)^* = B^+ C(B^+)^*$

We claim $X = \frac{1}{2}(D^*)^+ E + \frac{1}{2}(D^*)^+ E(I - D^+ D) + BB^+ (Z - Z^*)D$ is a general solution of equation (1). To do this substitute in the left side of operator equation $D^* XB^* (B^+)^* + B^+ BX^* D = E$ then we get:

$$D^* XB^* (B^+)^* + B^+ BX^* D = D^* \left(\frac{1}{2}(D^*)^+ E + \frac{1}{2}(D^*)^+ E(I - D^+ D) + BB^+ (Z - Z^*)D \right) B^* (B^+)^*$$
$$+ B^+ B \left(\frac{1}{2}(D^*)^+ E + \frac{1}{2}(D^*)^+ E(I - D^+ D) + BB^+ (Z - Z^*)D \right)^* D$$
$$= \frac{1}{2} D^* (D^+)^* EB^* (B^+)^* + \frac{1}{2} D^* (D^+)^* E(I - D^+ D)B^* (B^+)^* + D^* BB^+ (Z - Z^*)DB^* (B^+)^*$$
$$+ \frac{1}{2} B^+ BE^* D^+ D + \frac{1}{2} B^+ B(I - (D^+ D)^*)E^* D^+ D + B^+ BD^* (Z^* - Z)(B^+)^* B^* D$$

And by using the condition $C = C^*$ one can have:
$$= \frac{1}{2} D^* (D^+)^* EB^* (B^+)^* + \frac{1}{2} D^* (D^+)^* E(I - D^+ D)B^* (B^+)^* + D^* BB^+ (Z - Z^*)DB^* (B^+)^*$$
$$+ \frac{1}{2} B^+ BED^+ D + \frac{1}{2} B^+ B(I - D^+ D)ED^+ D + B^+ BD^* (Z^* - Z)(B^+)^* B^* D$$

$$= \frac{1}{2} D^* (D^+)^* EB^* (B^+)^* + \frac{1}{2} D^* (D^+)^* EB^* (B^+)^* - \frac{1}{2} D^* (D^+)^* ED^+ DB^* (B^+)^* + D^* BB^+ (Z - Z^*)DB^* (B^+)^*$$

$$+\frac{1}{2}B^+BED^+D+\frac{1}{2}B^+BED^+D-\frac{1}{2}B^+BD^+DED^+D+B^+BD^*(Z^*-Z)(B^+)^*B^*D$$

$$=D^*(D^+)^*DEB^*(B^+)^*+B^+BED^+D-\frac{1}{2}D^*(D^+)^*ED^+DB^*(B^+)^*-\frac{1}{2}B^+BD^*(D^+)^*ED^+D$$

$$+D^*BB^+(Z-Z^*)DB^*(B^+)^*+B^+BD^*(Z^*-Z)(B^+)^*B^*D$$

$$=D^+DE+ED^+D-\frac{1}{2}D^+DED^+D-\frac{1}{2}D^+DED^+D+D^*BB^+(Z-Z^*)DB^*(B^+)^*+B^+BD^*(Z^*-Z)(B^+)^*B^*D$$

$$=D^+DE+ED^+D-D^+DED^+D+D^*BB^+(Z-Z^*)DB^*(B^+)^*+B^+BD^*(Z^*-Z)(B^+)^*B^*D$$

$$=E+D^*BB^+(Z-Z^*)DB^*(B^+)^*+B^+BD^*(Z^*-Z)(B^+)^*B^*D$$

And by using the condition $A^*BB^+=BB^+A^*$, one can have:

$$=E+D^*(Z-Z^*)D+D^*(Z^*-Z)D$$

$$=E+D^*ZD-D^*Z^*D+D^*Z^*D-D^*ZD$$

$$=E+0$$

$$=E$$

Conversely, since $A^*XB^*+BX^*A=C$ so, $\left(A^*XB^*+BX^*A\right)^*=C^*$, therefore; $BX^*A+A^*XB^*=C^*$, then, $C=C^*$.

Also, $(I-D^+D)E(I-D^+D)=(I-D^+D)(D^*XB^*(B^+)^*+B^+BX^*D)(I-D^+D)$

$$=(I-D^+D)(D^*XB^*(B^+)^*+B^+BX^*D-D^*XB^*(B^+)^*D^+D-B^+BX^*D)$$

$$=(I-D^+D)(D^*XB^*(B^+)^*-D^*XB^*(B^+)^*D^+D)$$

$$=D^*XB^*(B^+)^*-D^*XB^*(B^+)^*D^+D-D^*XB^*(B^+)^*+D^*XB^*(B^+)^*D^+D$$

$$=0$$

2- Some properties of the mapping μ_A

Now, we give some properties of the map $\mu_{A,B}:B(H)\to B(H)$ is defined by $\mu_A(X)=A^*XB^*+BX^*A, X\in B(H)$ where A and B are known operator in

In the first we show this map μ_A is an additive map in fact

$$\mu_A(X_1+X_2)=A^*(X_1+X_2)B^*+B(X_1+X_2)^*A$$

$$A^*X_1B^*+A^*X_2B^*+BX_1{}^*A+BX_2{}^*A$$

$$(A^*X_1B^*+BX_1{}^*A)+(A^*X_2B^*+BX_2{}^*A)$$

$$=\mu_A(X_1)+\mu_A(X_2)$$

But the map μ_A is no homogenous

Since $\mu_A(\alpha X)=A^*(\alpha X)B^*+B(\alpha X)^*A$

$$=\alpha A^*XB^*+\alpha BX^*A$$

$$\neq\alpha\mu_A(X)$$

But if H is a Real Hilbert space then μ_A is a linear map

Now we give more properties of this map by the following proposition.

Proposition (2.1):
The map μ_a is a bounded
Proof:
$\|\mu_A(X)\|=\|A^*XB^*+BX^*A\|$
$\leq\|A^*XB^*\|+\|BX^*A\|$
$\leq\|A^*\|\|X\|\|B^*\|+\|B\|\|X^*\|\|A\|$
$=2\|A\|\|B\|\|X\|$
$=M\|X\|$
Where $M=2\|A\|\|B\|$ clearly, $M\geq 0$, Therefore the map μ_A is bounded
And the following proposition shows in general the map $\mu_A:B(H)\to B(H)$ is not necessary one to one
Now the following theorem study some properties of the Rang of the map μ_A

Theorem (2.2):
Let Range $(\mu_A)=\{A^*XB^*+BX^*A, X\in B(H)\}$then :-
i) $(Range(\mu_A))^*=Range(\mu_A)$
ii) $\alpha Range(\mu_A)=Range(\mu_A),\alpha\in R$
iii) iRange $(\mu_A)=\{AXB-BX^*A, X\in B(H)$
Proof:
i) $(Range\ \ (\mu_A))^*=\{(A^*XB^*+BX^*A)^*, X\in B(H)$
$=\{(BX^*A+A^*XB^*), X\in B(H)\}$
$=\{A^*XB^*+BX^*A, X\in B(H)\}$
$=Range(\mu_A)$
ii) $\alpha\ \ Range\ \ (\mu_A)=\{\alpha(A^*XB^*+BX^*A), X\in B(H)\}$
$=\{A^*(\alpha X)B^*+B(\alpha X)^*A, X\in B(H)\}$
$=\{A^*X_1B^*+BX_1^*A, X\in B(H)\} Where X_1=\alpha X$
$=Range(\mu_A)$
iii) $i\ \ Range\ \ (\mu_A)=\{(iA^*XB^*+iBX^*A), X\in B(H)\}$
$=\{(A^*(iX)B^*+B(iX)^*A), X\in B(H)\}$
$=\{A^*X_1B^*-BX_1^*A, X_1\in B(H)\} Where X_1=iX$

Theorem (2.3):
If $C_1,C_2\in Range(\mu_A). Then, C_1-C_2\in Range(\mu_A)$
Proof:
Let $C_1,C_2\in Range(\mu_A)$ then there exist $X_1$and $X_2\in B(H)$ such that
$A^*X_1B^*+BX_1^*A=C_1 and A^*X_2B^*+BX_2^*A=C_2$
$So, A^*(X_1-X_2)B^*+B(X_1-X_2)^*=C_1-C_2$

And therefore; $C_1 - C_2 \in Range(\mu_A)$

From above theorem one can have the following corollary.

Corollary (2.4):

If $C_1, C_2 \cdots, C_N \in Range(\mu_A)$.Then, $C_1 - C_2 - \cdots - C_N \in Range\ (\mu_A)$

Recall that a mapping f from a ring R into it self is called derivation if $f(ab) = af(b) + f(a)b \forall a, b \in R$. the following remark shows the mapping μ_A is not derivation .

Remark (2.5):

Since $\mu_{A,B}(XY) = A^*(XY)B^* + B(XY)^*A$

$$= A^*(XY)B^* + B(Y^*X^*)A$$

But $X\mu_{A,B}(Y) = XA^*YB^* + XBY^*A, and, \mu_{A,B}(X)Y = A^*XB^*Y + BX^*AY$

Thus , $\mu_{A,B}(XY) \neq X\mu_{A,B}(Y) + \mu_{A,B}(X)Y$, to illustrate this consider the following example .

Example (2.6):

Consider the unilateral shift operator U is defined on the Hilbert space ℓ^2={x=(x_1, x_2, …): $\sum_{i=1}^{\infty}|x_i| < \infty$, $x_i \in ¢$} by U(x_1,x_2,…)=(0,x_1,x_2,…), (x_1,x_2,…)$\in \ell^2$, then bilateral shift operator is the adjoint operator of U is U^*(x_1,x_2,…) =(x_2,x_3,…), (x_1,x_2,…)$\in \ell^2$.

Then $\mu_U(X) = BX + X^*A$ where B is the bilateral shift operator thus

$$\mu_U(U) = BU + BU = 2BU$$

$$But, \mu_U(U) + \mu_U(I)U = BU + BU + (B+U)U$$

$$= BU + BU + BU + U^2$$

$$= 3BU + U^2$$

$$\mu_U(U) \neq \mu_U(U) + \mu_U(I)U$$

Where I is identity operator therefore; μ_A is not derivation.

References:

[1] Dragan S. , "the solution of some operator equations", j. of the Korean math. Soci. Vol. No. 5, pp.1417-1425,2008.

[2] Djordjevic D. , "explicit solution of some operator equation $A^*X + X^*A = B$", J. compu. Appl. Math. , vol. 200, pp.701-704,2007

[3] Emad A., "the natural of the solution for the generalized lyapunov equations" J. AL-qadisiah for pure science vol.11, No.4, pp.323-333, 2006

[4] Erwen K. ,"introduction functional analysis", hon wiley and sons inc.,1978 .

المستخلص

في هذا البحث قدمنا الحلول العامة لمعادلة المؤثر المقيدة $A^*XB^* + BX^*A = C$ حيث A و B مؤثرات ليس لها معكوس والمعرفة على فضاء هلبرت المعقدة H وكذلك خواص التطبيق $\mu_{A,B}(X) = A^*XB^* + BX^*A$.