On Z-open sets

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Abstract

Continuous real valued function is an important tool in topology. In this paper, real valued continuous functions are used to define some kind of sets called zero sets which is the inverse image of zero of a real valued function from a topological space, the complement of zero sets is called a cozero sets, and the family of all cozero sets of a topological space X forms a base for a topology on X which is called the Z-topology on X and its elements is called Z-open sets.

In this work we study the properties of these sets with some relations between it and other sets like open, cozero and zero sets. On the other hand we proved some results and characterizations with some examples.

Key words: Zero sets, cozero sets, continuous real valued functions.

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1- Introduction

In this paper the concepts of zero set cozero set and Z-open set are given with some examples to explain these sets, also important definitions, remarks and properties about these sets are given.

The family of functions from a topological space X into R is denoted by R^X and the family of all continuous real valued functions on a topological space X is denoted by C(X).

i.e. $C(X) = \{ f : X \longrightarrow \mathbb{R} \text{ continuous}\}$.

It is known that C(X) is a ring with + and where $(f + g)(x) = f(x) + g(x)$ and $(f \cdot g)(x) = f(x) \cdot g(x)$, [2].

2- Definition [4]

Let X be a topological space and let A be a subset of X, then A is called a zero set in X if there exists a continuous function $f : X \longrightarrow \mathbb{R}$ such that $f^{-1}(0) = A$. The family of all zero sets of X is denoted by Z(X).

i.e. $Z(f) = \{x \in X: f(x) = 0\}$, $f \in C(X)$ and $Z(X) = \{Z(f): f \in C(X)\}$.

The complement of the zero set is called a cozero set [2]. The family of all cozero sets is denoted by coz(X).

3- Proposition [2]

(i) $Z(f) = Z(|f|) = Z(f^n)$ for all $n \in N$.

(ii) $Z(0) = X$ and $Z(1) = \phi$.

(iii) $Z(f \cdot g) = Z(f) \cup Z(g)$.
(iv) $Z(f + g) = Z(|f| + |g|) = Z(f) \cap Z(g)$.

4- Remark [2]

Every zero set is a closed set.
On the other side every cozero set is open set.

The converse of remark (4) is not true in general, see the following examples.

4.1 Example

Let $X = \{1,2,3\}$
$\tau = \{X,\phi,\{1\},\{2\},[1,2\}\}$
The family of zero sets of X is $\{X,\phi\}$ is this example $\{2,3\}$ is closed set but it is not a zero set.

4.2 Example [5]

Let $X = (0,1)$.
$\tau = \{u_n = (0, 1-\frac{1}{n}): n = 2,3,4,\ldots\} \cup \{X,\phi\}$.
The family of zero sets of X is $\{X,\phi\}$ is this example $(1,\frac{1}{2})$ is open set in X but not cozero set.

5- Remark

The intersection of any family of zero sets need not be a zero set.

In other words the union of any family of cozero sets need not be a cozero set.
For example:

Let $X = \mathbb{R}$ the set of reals, we define a base B for τ on X as follows:
$u \in B$ if $u = \{r\}$ where $r \in \mathbb{R}$ and $r \neq 0$ or $0 \in u$ and u^c is countable implies $\{0\}$ is closed set since $\mathbb{R}\backslash\{0\}$ is open set but $\{0\}$ is not a zero set.
Since $\nexists\, f : (\mathbb{R},\tau) \longrightarrow (\mathbb{R},\tau_u)$ continuous function such that $f^{-1}(0) = \{0\}$ where τ_u is the usual topology on $\mathbb{R}$.
If there exists $f : \mathbb{R} \longrightarrow \mathbb{R}$ continuous such that $f^{-1}(0) = \{0\}$ then for $(a,b) \in \tau_u$ we get $f^{-1}(a,b) \in \tau$.
If $0 \in (a,b)$ and $f(x) \notin (a,b)\ \forall\ x \in \mathbb{R}$ then $f^{-1}(a,b) = \{0\}$ is open set which is a contradiction. Let $A_r = \mathbb{R}\backslash\{r\}$ where $r \in \mathbb{R}\backslash\{0\}$, then A_r is a zero set $\forall\ r \in \mathbb{R}\backslash\{0\}$.
Since let $f : (\mathbb{R},\tau) \longrightarrow (\mathbb{R},\tau_u)$ such that

$$f(x)=\begin{cases}0 & \text{if } x\neq r\\ c\neq 0 & \text{if } x=r\end{cases}$$

Then f is continuous since

$$f^{-1}((a,b))=\begin{cases}\mathbb{R} & \text{if } 0,c\in(a,b)\\ \varphi & \text{if } 0,c\notin(a,b)\\ \{r\} & \text{if } c\in(a,b)\text{ and } 0\notin(a,b)\\ \mathbb{R}\setminus\{r\} & \text{if } c\notin(a,b)\text{ and } 0\in(a,b)\end{cases}$$

But $\bigcap_{r\in\mathbb{R}\setminus\{0\}} A_r=\{0\}$ which is not a zero set. So $\bigcup_{r\in\mathbb{R}\setminus\{0\}} A_r^c$ is not a cozero set.

6- Remark

Tha family coz(X) is not a topology on X (see the example of remark (5)).

7- Definition [7]

Let X be a topological space, let coz(X) be the family of cozero sets on X. The topology τ_Z for which the family coz(X) is a base is called the Z-topology on X, and the members of τ_Z is called Z-open sets.

8- Proposition

Let X be a topological space, then the family of Z-open set is a topology on X.

Proof:

(i) X and ϕ are Z-open sets since they are cozero sets.

(ii) If A and B are Z-open sets then $A\cap B$ is a Z-open set since $A\cap B$ is a cozero set, [2].

(iii) Let $\{A_\alpha\}_{\alpha\in\Lambda}$ be a family of Z-open sets then $\cup A_\alpha$ is Z-open set by definition of Z-open set. ■

9- Remark

In the following examples we calculate τ_Z in some known topological spaces.

Examples:

(i) If X is the indiscrete space, then $\tau_Z = I = \{X,\phi\}$.

(ii) If X is the discrete space, then $\tau_Z = D = \mathbb{P}(X)$.

(iii) The family of Z-open sets of real numbers with the usual topology τ_u is τ_u.

(iv) Let X be any infinite set and let τ_{co} be the cofinite topology on X, then the family of Z-open sets in (X, τ_{co}) is $\{X,\phi\} = I$.

10- Remark [2]

Every zero set is G_σ, which is a countable intersection of open sets.

The converse of remark (10) is not true in general, for example {1} is G_σ set in example 4.1, which is not a zero set.

The following proposition was written in [2] without proof and we prove it.

11- Proposition

In any normal space. Every closed G_σ set is a zero set.

Proof:

Let A be a closed G_σ set to show A is a zero set. Which means to show that there exists a continuous function $f : X \longrightarrow \mathbb{R}$ such that $f^{-1}(0)$ = A.

$A = \bigcap_{n=1}^{\infty} A_n$ where A_n is open set, $n \in N$. Then $A \subseteq A_n$, $\forall\ n \in N$, then $A_n^c \cap A = \phi$. But A_n^c is closed set, $\forall\ n \in N$ since A_n is open set and A is closed set (given) so by using (Urysohns lemma) there exists $f \in C(X)$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = A_n^c$ then A is a zero set. ■

12- Remark

The condition that is given in proposition (11) which says that X is normal is a necessary condition, as shown in the following example.

Example:

Take the set of natural numbers with cofinite topology (N,τ_{co}). This space is not normal and $\tau_Z = \{N,\phi\}$. Let A = {1}, then A is a closed and G_σ set since $A = \bigcap_{n=2}^{\infty} A_n$ where $A_n = N\backslash\{n\}$; $n \in N\backslash\{1\}$, but it is not a zero set.

13- Remark

It is clear that every open set is G_σ. But the converse is not true in general, for example:

In $(\mathbb{R},\tau_u)$ the family $\{(\frac{-1}{n},\frac{1}{n}); n \in N\}$ is a countable family of open sets and $\bigcap_{n=1}^{\infty}(\frac{-1}{n},\frac{1}{n}) = \{0\}$ is G_σ set, which is not open set.

We give a characterization for Z-open sets in the following proposition.

14- Proposition

Let X be a topological space. If the family of open sets τ equals the family of closed sets $\mathcal{F}$ then every open set is a cozero set.

Proof:

Let U be an open set then U^c is closed set since $\tau = \mathcal{F}$, which means that X is a normal space and U^c is closed G_σ set in X then U^c is zero set implies that U is cozero set. ■

15- Definition [5]

Let X be a topological space. X is said to be perfectly normal space if it is normal space and every closed set is G_σ set.

The following proposition was written in [3] without proof.

16- Proposition

A topological space is perfectly normal if and only if every closed set is a zero set.

Proof:

For the "if" direction

Suppose that X is perfectly normal, to show that every closed subset of X is a zero set.

Let A be a closed subset of X, so A is a G_σ set (given), hence A is a zero set in X by proposition (11).

Now for 'only if" direction.

Suppose the every closed subset of X is a zero set to show X is perfectly normal space.

i.e. X is normal and every closed subset of X is a G_σ subset.

Let A be a closed subset of X, then A is a G_σ set since A is a zero set and every zero set is a G_σ set by remark (10).

To show X is normal, let F and E be two closed disjoint subsets of X, then F and E are zero sets (given) implies there are two continuous function f and g such that $f, g : X \longrightarrow \mathbb{R}$ and $f^{-1}(0) = F$ while $g^{-1}(0) = E$.

Define $h : X \longrightarrow \mathbb{R}$ such that

$$h(x) = \begin{cases} |\sin f(x)| & \text{if } x \in F \\ |\cos g(x)| & \text{if } x \notin F \end{cases}$$

h is continuous function since f and g are continuous functions moreover $h(x) = 0 \ \forall \ x \in F$ and $h(x) = 1 \ \forall \ x \in E$, since $0 \leq |\sin f(x)| \leq 1$ and $0 \leq |\cos g(x)| \leq 1$, $h : X \longrightarrow [0,1]$ is a continuous function with $h(F) = 0$ and $h(E) = 1$, hence by Urysohn's lemma X is normal. ■

17- Definition [6]

A topological space X is said to be completely regular if $\forall \ x \in X$, and for each closed set F such that $x \notin F$ then there exists $f : X \longrightarrow [0,1]$ which is continuous such that $f(x) = 0$ and $f(F) = 1$.

The following proposition was written in [6] without proof.

18- Proposition

Any topological space (X,τ) is completely regular iff the cozero sets forms a base for τ.

Proof:

The "if" direction.

Suppose X is completely regular, to show that coz(X) is a base for τ.
Let $U \in \tau$ and let $y \in U$, so $y \notin U^c$ which is closed subset of X, implies there is a continuous function $f : X \longrightarrow [0,1]$ such that $f(y) = 0$ and $f(U^c) = 1$ since X is completely regular (given).
Now define $g: X \longrightarrow [0,1]$ such that $g(x) = 1 - f(x)$, then it is clear that g is continuous and $g(y) = 1$ while $g(U^c) = 0$, let $A = Z(g)$, so $U^c \subseteq A$ and hence $A^c \subseteq U$, but A^c is a cozero set and $y \in A^c$ since $y \notin A$ for $g(y) \neq 0$ which means that we have a cozero set contains y and is contained in U for each y in U, so U is a union of cozero sets, i.e. coz(X) is a base for τ.

To prove the "only if" direction.

Suppose coz(X) is a base for τ, to show X is completely regular, let F be a closed subset of X and $y \in X$ such that $y \notin F$, then $y \in F^c$ which an open set of X, but coz(X) is a base for τ, so there is a cozero set A in X such that $y \in A \subseteq F^c$ and a continuous function $f : X \longrightarrow \mathbb{R}$ such that $f^{-1}(0) = A^c$ and $f(y) = c;\ c \neq 0$ since $y \notin A^c$ while $f(F) = 0$ since $F \subseteq A^c$.
Now define $g : X \longrightarrow [0,1]$ such that $g(x) = 1 - \left|\frac{\sin f(x)}{\sin c}\right|$, $g(F) = 1$ and $g(y) = 0$ where g is continuous, so X is completely regular. ■

We give the following corollary.

19- Corollary

Let (X,τ) be a completely regular space, and let U be a subset of X, then U is open set iff U is Z-open set.

Proof:

Suppose U is open set, to show U is Z-open set.
Since X is completely regular then by proposition (18) the cozero sets is a base for τ, then every open set is a union of a cozero sets, which means that U is a Z-open set by definition of Z-open set.

The converse is true always. ■

20- Definition [6]

A topological space X is said to be hyperconnected space if every two open sets in X have a nonempty intersection.

The following proposition was written in [5] without proof.

21- Proposition

X is a hyperconnected space iff every non empty open set of X is dense in X.

Proof:

Suppose X is hyperconnected space, to show every open set of X is dense, let U be a non empty open subset of X, to show U is dense i.e. cl U = X. It is enough to show X $\subseteq$ cl U since cl U $\subseteq$ X for each U in X, let x $\in$ X and V be an open subset of X such that x $\in$ V then U $\cap$ V $\neq \phi$, implies x $\in$ cl U then X $\subseteq$ cl U. So, cl U = X.

Now, to show X is hyperconnected, let U, V be open non empty subsets of X to show U $\cap$ V $\neq \phi$. Suppose U $\cap$ V = ϕ implies if x $\in$ V, then x $\notin$ cl U this is a contradiction since cl U = X, then X is hyperconnected space. ■

The following proposition was written in [5] without proof.

22- Proposition

Every continuous real valued function on a hyperconnected space is constant.

Proof:

Let X be a hyperconnected space to show that every continuous real valued function is constant. Suppose that there exists f : X $\longrightarrow \mathbb{R}$ continuous which is not constant. Let x, y $\in$ X such that f(x) $\neq$ f(y) and let $U=(f(x)-\frac{\varepsilon}{2}, f(x)+\frac{\varepsilon}{2})$, $V=(f(y)-\frac{\varepsilon}{2}, f(y)+\frac{\varepsilon}{2})$ where $\varepsilon = |f(x) - f(y)|$, U $\cap$ V = ϕ, f^{-1}(U $\cap$ V) = ϕ, f^{-1}(U) $\cap$ f^{-1}(V) = ϕ, where f^{-1}(U) $\neq \phi$ since x $\in$ f^{-1}(U) and f^{-1}(V) $\neq \phi$ since y $\in$ f^{-1}(V), this is a contradiction since X is hyperconnected space. ■

23- Corollary

If X is hyperconnected space then τ_Z = I.

Proof: clear.

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حول المجموعات المفتوحة من النمط – Z *

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المستخلص:

تعتبر الدوال المستمرة ذات القيم الحقيقية اداة ضرورية في التبولوجيا. في هذا البحث تم استخدام هذه الدوال من أجل تعريف نمط معين من المجموعات تسمى المجموعات الصفرية وهي مجموعة العناصر في فضاء التبولوجيا التي صورتها صفر بالنسبة لدالة مستمرة ذات قيم حقيقية ويقال لمتممات المجموعة الصفرية بأنها متممة صفرية. وتمثل عائلة متممات المجموعة الصفرية أساساً لتبولوجيا على المجموعة تسمى التبولوجيا من النمط – Z، كما تسمى عناصر هذه التبولوجيا المجموعة المفتوحة من النمط – Z.

في هذا العمل قمنا بدراسة خواص تلك المجموعات وعلاقاتها مع المجموعات المفتوحة والمجموعات الصفرية ومتممات الصفرية.

من ناحية أخرى قمنا ببرهنة بعض النتائج كما أعطينا بعض التصنيفات والامثلة.

كلمات مفتاحية: المجموعة الصفرية، متممة المجموعة الصفرية، الدوال المستمرة ذات القيم الحقيقية.