Some Properties Of Cartesian Product Of Two Fuzzy Normed Spaces

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Abstract

In this paper , the concept of the Cartesian Product of two fuzzy normed spaces is presented. Some basic properties and theorems on this concept are proved. The main goal of this paper is to prove that the Cartesian product of two complete fuzzy normed spaces is a complete fuzzy normed space.

Key words:Fuzzy normed space , Cartesian product , Cauchy sequence , complete fuzzy normed space.

1- Introduction

The fuzzy set concepts was introduced in mathematics by K.Menger in 1942 and reintroduced in the system theory by L.A.Zadeh in 1965.
In 1984, Katsaras [1] , first introduced the notation of fuzzy norm on linear space, in the same year Wu and Fang [4] also introduced a notion of fuzzy normed space . Later on many other mathematicians like Felbin [2] , Cheng and Mordeson [10] , Bag and Samanta [12], J.Xiao and X.Zhu [8,9] , Krishna and Sarma [11] , Balopoulos and Papadopoulos [13] etc, have given different definitions of fuzzy normed spaces .
J.Kider introduced the definition of fuzzy normed space[7] , we use this definition to prove that the Cartesian product of two fuzzy normed spaces is also fuzzy normed space.

The structure of the paper is as follow : In section 2 we present some fundamental concepts . In section 3, the definition of fuzzy normed space appeared [7] is used to prove that the cartesain product of two fuzzy normed spaces is also fuzzy normed space, then we prove that the cartesain product of two complete fuzzy normed spaces is complete fuzzy normed space.

2. Preliminaries

In this section, we briefly recall some definitions and preliminary results which are used in this paper.

Definition 2.1 [7]:

Let U be a vector space over field **K** (**K**=R or **K**=C) . Put I=[0,1] then $\tilde{N}: U \times I \rightarrow I$ is said to be a fuzzy norm on U in case for each $u, v \in U$ and $\lambda \in K$ the following conditions hold

(N1) if $\alpha = 0$ then $\tilde{N}(u, \alpha) = 0$.

(N2) if $\alpha \neq 0$ then $\tilde{N}(u, \alpha) = 0$ if and only if $u = 0$.

(N3) $\tilde{N}(\lambda u, \alpha) = |\lambda| \tilde{N}(u, \alpha)$.

(N4) $\tilde{N}(u + v, \alpha) \leq \tilde{N}(u, \alpha) + \tilde{N}(v, \alpha)$.

(N5) if $0 < \sigma \leq \alpha < 1$ then $\tilde{N}(u, \alpha) \leq \tilde{N}(u, \sigma)$ and there exists $0 < \alpha_n < \alpha$ such that $\lim_{n\rightarrow\infty} \tilde{N}(u, \alpha_n) = \tilde{N}(u, \alpha)$.

Then $\tilde{N}$ is called fuzzy norm and $(U, \tilde{N})$ is called fuzzy normed space.

Proposition 2.2 [7]:

Let $(U, \|.\|)$ be an ordinary normed space , define $\tilde{N}(u, \alpha) = \frac{1}{\alpha} \|u\|$ for $\alpha > 0$ and $\tilde{N}(u, \alpha) = 0$ for $\alpha = 0$. Then $(U, \tilde{N})$ is a fuzzy normed space.

Example 2.3 [7]:

Let $U = R$, then $\tilde{N}(u, \alpha) = \frac{1}{\alpha} |u|$ is a fuzzy norm on R by proposition 2.2 called the usual fuzzy norm.

Definition 2.4 [6]:

Let A and B be any two sets ,the Cartesian product is denoted by $A \times B$ and is defined by $A \times B = \{(a, b) | a \in A , b \in B\}$.

Definition 2.5 [3]:

Let U be a universe. A fuzzy set X over U is a set defined by afunction μ_x representing a mapping

$\mu_x : U \rightarrow [0,1]$

μ_x is called a membership function of X ,and the value $\mu_x(u)$ is called the grade of membership of u$\in U$. Thus a fuzzy set X over U can be represented as follows

$$X = \{(\mu_x(u)/u) : u \in U, \mu_x(u) \in [0,1]\}$$

Definition 2.6 [5]:

A fuzzy set x_α of a set S is called a fuzzy point if

$$x_\alpha(y) = \begin{cases} \alpha & if\ x = y \\ 0 & otherwise \end{cases} .$$

for all $x, y \in S$ and $\alpha \in (0,1]$.

Definition 2.7 [7]:

A sequence $\{(u_n, \alpha_n)\}$ of fuzzy points in a fuzzy normed space $(U, \tilde{N})$ converges to $u_\alpha \in U$ if $\lim_{n\to\infty} \tilde{N}(u_n - u, \lambda) = 0$, where $\alpha, \alpha_n \in (0,1]$ and $\lambda = min\{\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n, ..\}$.

Definition 2.8 [7]:

A sequence $\{(u_n, \alpha_n)\}$ of fuzzy points in a fuzzy normed space $(U, \tilde{N})$ is Cauchy if for any $\varepsilon > 0$ there is integer $n_\circ > 0$ such that for all $m, n > n_\circ$, we have

$\tilde{N}(u_m - u_n, \lambda) < \varepsilon$ where $\lambda = \min\{\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots\}$.

3- Cartesian product of two fuzzy normed spaces

In this section , we introduce the definition of the Cartesian product of two fuzzy normed spaces , then we prove that the Cartesian product of two fuzzy normed spaces is also fuzzy normed space.

Finally , we prove the completeness of the Cartesian product of two complete fuzzy normed spaces.

Definition 3.1:

Let $(U, \widetilde{N_1})$ and $(V, \widetilde{N_2})$ be two fuzzy normed spaces .The Cartesian product of $(U, \widetilde{N_1})$ and $(V, \widetilde{N_2})$ is the product space $(U \times V, \tilde{N})$ where $U \times V$ is the Cartesian product of the sets U and V and $\tilde{N}$ is a mapping from $(U \times V \times [0,1])$ into $[0,1]$ given by

$\tilde{N}((u,v),\alpha) = \widetilde{N_1}(u,\alpha) + \widetilde{N_2}(v,\alpha)$, for all $(u,v) \in U \times V$, $\alpha \in (0,1)$.

Theorem 3.2:

Let $(U, \widetilde{N_1})$ and $(V, \widetilde{N_2})$ be two fuzzy normed spaces then $(U \times V, \tilde{N})$ is a fuzzy normed space .

Proof:

Let $(u,v) \in U \times V$ and $\lambda \in K$

(N1) if $\alpha = 0$ then $\widetilde{N_1}(u,\alpha) = 0$ and $\widetilde{N_2}(v,\alpha) = 0$ so

$\widetilde{N_1}(u,\alpha) + \widetilde{N_2}(v,\alpha) = 0$ which implies that $\tilde{N}((u,v),\alpha) = 0$.

(N2) if $\alpha \neq 0$ then $\tilde{N}((u,v),\alpha) = 0 \leftrightarrow \widetilde{N_1}(u,\alpha) + \widetilde{N_2}(v,\alpha) = 0.$

$\leftrightarrow \widetilde{N_1}(u,\alpha) = 0$ and $\widetilde{N_2}(v,\alpha) = 0$.

$\leftrightarrow (u,\alpha) = 0$ and $(v,\alpha) = 0$.

$\leftrightarrow u = 0\ and\ v = 0$.

$\leftrightarrow (u,v) = (0,0)$.

(N3) $\tilde{N}(\lambda(u,v),\alpha) = \widetilde{N_1}(\lambda u.\alpha) + \widetilde{N_2}(\lambda v,\alpha)$

$= \lambda\widetilde{N_1}(u.\alpha) + \lambda\widetilde{N_2}(v,\alpha)$

$= \lambda[\widetilde{N_1}(u,\alpha) + \widetilde{N_2}(v,\alpha)]$

$= \lambda \tilde{N}((u,v),\alpha)$.

(N4) $\tilde{N}\left(\left((u,v)+(u_1,v_1)\right),\alpha\right) \le \tilde{N}((u+u_1\ ,v+v_1),\alpha)$

$\le \tilde{N_1}(u+u_1,\alpha)+\tilde{N_2}(v+v_1,\alpha)$.

$\le \tilde{N_1}(u,\alpha)+\tilde{N_1}(u_1,\alpha)+\tilde{N_2}(v,\alpha)+\tilde{N_2}(v_1,\alpha)$.

$\le \left(\tilde{N_1}(u,\alpha)+\tilde{N_2}(v,\alpha)\right)+\left(\tilde{N_1}(u_1,\alpha)+\tilde{N_2}(v_1,\alpha)\right)$.

$= \tilde{N}\left((u,v),\alpha\right)+\tilde{N}((u_1,v_1),\alpha)$.

(N5) if $0<\sigma\le\alpha<1$ then $\tilde{N_1}(u,\alpha)\le\tilde{N_1}(u,\sigma)$ and $\tilde{N_2}(v,\alpha)\le \tilde{N_2}(v,\sigma)$, so $\tilde{N}\left((u,v),\alpha\right)\le\tilde{N}((u,v),\sigma)$ and there exist

$0<\alpha_n<\alpha$ such that $\lim_{n\to\infty}\tilde{N_1}(u,\alpha_n)=\tilde{N_1}(u,\alpha)$ and $\lim_{n\to\infty}\tilde{N_2}(v,\alpha_n)=\tilde{N}(v,\alpha)$ which implies that $\lim_{n\to\infty}\tilde{N}\left((u,v),\alpha_n\right)=\tilde{N}((u,v),\alpha)$.

Thus $\left(U\times V\ ,\tilde{N}\right)$ is fuzzy normed space.

<u>Proposition 3.3</u>

If $\{(u_n,\alpha_n)\}$ is a sequence of fuzzy points in the fuzzy normed space $(U,\tilde{N_1})$ converges to u_α in U and $\{(v_n,\alpha_n)\}$ is a sequence of fuzzy points in the fuzzy normed space $(V,\tilde{N_2})$ converges to v_α in V then $\{\left((u_n,v_n),\alpha_n\right)\}$ is a sequence in $U\times V$ converges to (u_α,v_α) in $\left((U\times V),\tilde{N}\right)$ where $\alpha=\min\{\alpha_n \mid n\in N\}$.

Proof:

To conclude that sequence $\{\left((u_n,v_n),\alpha_n\right)\}$ in $U\times V$ converges to (u_α,v_α) we show that $\lim_{n\to\infty}\tilde{N}\left((u_n,v_n)-(u,v),\lambda\right)=0$.

By theorem 3.2 , $(U\times V,\tilde{N})$ is a fuzzy normed space . since $(u_\alpha,\alpha_n)\to u_\alpha$. and $(v_\alpha,\alpha_n)\to v_\alpha$ so $\lim_{n\to\infty}\tilde{N_1}(u_n-u,\lambda_1)=0$ and $\lim_{n\to\infty}\tilde{N_2}(v_n-v,\lambda_2)=0$.

So $\lim_{n\to\infty}\tilde{N}\left((u_n,v_n)-(u,v),\lambda\right)=$

$\lim_{n\to\infty}\tilde{N_1}(u_n-u,\lambda)+\lim_{n\to\infty}\tilde{N_2}(v_n-v\ ,\lambda)=0+0=0$

Thus $\{\left((u_n,v_n),\alpha_n\right)\}$ converges to (u_α,v_α).

<u>Proposition 3.4</u>

If $\{(u_n,\alpha_n)\}$ is Cauchy sequence in $(U,\tilde{N_1})$ and $\{(v_n,\alpha_n)\}$ is Cauchy sequence in $(V,\tilde{N_2})$ then $\{(u_n,v_n),\alpha_n\}$ is Cauchy sequence in $(U\times V\ ,\tilde{N})$.

Proof:

By theorem 3.2 , $(U \times V , \widetilde{N})$ is a fuzzy normed space . since $\{(u_n, \alpha_n)\}$ and $\{(v_n, \alpha_n)\}$ are Cauchy sequences then for each given $\varepsilon > 0$ there is a positive constant $n_\circ$ such that ,

$\widetilde{N_1}(u_m - u_n, \lambda_1) < \frac{\varepsilon}{2}$ and $\widetilde{N_2}(v_m - v_n, \lambda_2) < \frac{\varepsilon}{2}$ for every $m, n > n_\circ$

Now for each $m, n > n_\circ$

$$\widetilde{N}((u_m, v_m) - (u_n, v_n), \lambda) = \widetilde{N}((u_m - u_n , v_m - v_n), \lambda).$$
$$= \widetilde{N_1}(u_m - u_n, \lambda_1) + \widetilde{N_2}(v_m - v_n, \lambda_2).$$
$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

Thus $\{(u_n, v_n), \alpha_n\}$ is Cauchy sequence in $(U \times V , \widetilde{N})$.

Definition 3.5:

Let $(U , \widetilde{N})$ be a fuzzy normed space , then a fuzzy normed space in which every fuzzy Cauchy sequence $\{(u_n, \alpha_n)\}$ is convergent is said to be complete.

Theorem 3.6

If $(U, \widetilde{N_1})$ and $(V, \widetilde{N_2})$ are complete fuzzy normed spaces then Cartesian product $(U \times V , \widetilde{N})$ is a complete fuzzy normed space .

Proof:

Let $\{(u_n, v_n), \alpha_n\}$ be a Cauchy sequence in $U \times V$ that is for any given $\varepsilon > 0$ there is $n_\circ$ such that $\widetilde{N}\left(\left((u_m, v_m) - (u_n, v_n)\right), \lambda\right) < \varepsilon$ which implies that

$\widetilde{N_1}(u_m - u_n, \lambda_1) + \widetilde{N_2}(v_m - v_n , \lambda_2) < \varepsilon$ where $\lambda = \min\{\lambda_1, \lambda_2\}$

so that $\widetilde{N_1}(u_m - u_n, \lambda_1) < \varepsilon$ and $\widetilde{N_2}(v_m - v_n, \lambda_2) < \varepsilon$ that is $\{(u_n, \alpha_n)\}$ is Cauchy in $(U, \widetilde{N_1})$ and $\{(v_n, \alpha_n)\}$ is Cauchy in $(V, \widetilde{N_2})$, but $(U, \widetilde{N_1})$ and $(V, \widetilde{N_2})$ are complete fuzzy normed spaces , so there is u_α in U and v_α in V such that $\{(u_n, \alpha_n)\}$ converges to u_α and $\{(v_n, \alpha_n)\}$ converges to v_α that is $\lim_{n\to\infty} \widetilde{N_1}(u_n - u, \lambda_1) = 0$ and $\lim_{n\to\infty} \widetilde{N_2}(v_n - v, \lambda_2) = 0$.

Now ,

$\lim_{n\to\infty} \widetilde{N}\left(\left((u_n, v_n) - (u, v)\right), \lambda\right) =$

$\lim_{n\to\infty} \widetilde{N_1}(u_n - u, \lambda_1) + \lim_{n\to\infty} \widetilde{N_2}(v_n - v, \lambda_2) = 0+0 =0$

Thus $\{(u_n, v_n), \alpha_n\}$ converges to (u_α, v_α) in $U \times V$, therefore $(U \times V , \widetilde{N})$ is a complete fuzzy normed space.

Theorem 3.7

If $(U \times V, \tilde{N})$ is a fuzzy normed space , then $(U, \widetilde{N_1})$ and $(V, \widetilde{N_2})$ are fuzzy normed spaces by defining $\widetilde{N_1}(u, \alpha) = \tilde{N}((u, 0), \alpha)$ and $\widetilde{N_2}(v, \alpha) = \tilde{N}((0, v), \alpha)$.

Proof:

Let $u \in U$ and $\gamma \in K$

(N1) if $\alpha = 0$ then $\tilde{N}((u, 0), \alpha) = 0 \to \widetilde{N_1}(u, \alpha) = 0$

(N2) if $\alpha \neq 0$ then $\widetilde{N_1}(u, \alpha) = 0 \leftrightarrow \tilde{N}((u, 0), \alpha) = 0$

$$\leftrightarrow (u, 0) = 0$$
$$\leftrightarrow u = 0$$

(N3) $\widetilde{N_1}(\gamma u, \alpha) = \tilde{N}((\gamma u, 0), \alpha) = |\gamma|\, \tilde{N}((u, 0), \alpha) = |\gamma|\, \widetilde{N_1}(u, \alpha)$.

(N4) $\widetilde{N_1}(u + u_1, \alpha) = \tilde{N}((u + u_1, 0), \alpha)$.

$$\leq \tilde{N}\big((u, 0), \alpha\big) + \tilde{N}((u_1, 0), \alpha).$$
$$= \widetilde{N_1}(u, \alpha) + \widetilde{N_1}(u_1, \alpha).$$

(N5) if $0 < \sigma \leq \alpha < 1$ then $\tilde{N}((u, 0), \alpha) \leq \tilde{N}((u, 0), \sigma)$ that is $\widetilde{N_1}(u, \alpha) \leq \widetilde{N_1}(u, \sigma)$.

Then there exists $0 < \alpha_n < \alpha$ such that $\lim_{n \to \infty} \widetilde{N_1}(u, \alpha_n) = \widetilde{N_1}(u, \alpha)$.

Thus $(U, \widetilde{N_1})$ is a fuzzy normed space.

Similarly we can prove that $(V, \widetilde{N_2})$ is a fuzzy normed space.

Theorem 3.8:

If $(U \times V, \tilde{N})$ is a complete fuzzy normed space , then $(U, \widetilde{N_1})$ and $(V, \widetilde{N_2})$ are complete fuzzy normed spaces .

proof:

$(U, \widetilde{N_1})$ and $(V, \widetilde{N_2})$ are fuzzy normed spaces by theorem 3.7 .

Let $\{(u_n, \alpha_n)\}$ be a Cauchy sequence in $(U, \widetilde{N_1})$ then $\{(u_n, 0), \alpha_n)\}$ is a Cauchy sequence in $U \times V$. But $U \times V$ is complete fuzzy normed space , that is there is $(u_\alpha, 0)$ in $U \times V$ such that $\{((u_n, 0), \alpha_n))\}$ converges to $(u_\alpha, 0)$.

Now , $\lim_{n \to \infty} \widetilde{N_1}(u_n - u, \alpha) = \lim_{n \to \infty} \tilde{N}((u_n - u, 0), \alpha) = 0$

That is $(U, \widetilde{N_1})$ is a complete fuzzy normed space. Similarly we can prove that $(V, \widetilde{N_2})$ is a complete fuzzy normed space.

References

[1] A.K. Katsaras , " Fuzzy topological vector spaces" II,FS, 12(1984) 143- 154.

[2] C.Felbin ," Finite dimensional fuzzy normed linear space" , Fuzzy sets and systems , 48 (1992) 239-248.

[3] C. Naim and C.Filiz , "Fuzzy parameterized fuzzy soft set theory and its applications" , (2010), 21-35.

[4] Congxin Wu, Jinxuan Faing , "Fuzzy generalization of Klomogoroffs theorem" ,J.Harbin Inst.Technol , 1(1984)1-7.

[5] E.H. Hamouda , "On Some Ideals of Fuzzy Points Semigroups", Gen.Math.Notes , 17(2013) 76-80 .

[6] Istratescu, V. and Vaduva, I. ," Product of Statistical Metric Spaces" , Acad.R.P.Roumaine Stud . Cerc.Math, 12(1961) 567-574.

[7] Jehad R.Kider , "New Fuzzy Normed Spaces", J.Baghdad Sci , vol. 9(2012) 559-564.

[8] J.Xiao and X.Zhu ," On linearly topological structure and property of fuzzy normed linear space" , Fuzzy sets and systems , 125(2002) 153-161.

[9] J.Xiao and X.Zhu, "Fuzzy normed space of operators and its completeness" , Fuzzy sets and systems , 133(2003) 389-399.

[10] S.C.Cheng , J.N.Mordeson , "Fuzzy linear operator and fuzzy normed linear spaces". Bull.Cal.Math.Soc . 86(1994) 429-436.

[11] S.V.Krishna, K.K.M.Sarma ," Separation of fuzzy normed linear spaces ", Fuzzy sets and systems , 63(1994)207-217.

[12] T.Bag and S.K. Samanta , "Fuzzy bounded linear operated in Felbin's type fuzzy normed linear space ", Fuzzy sets and systems ,159(2008) 685-707.

[13] V. Balopoulos and B.K. Papadopoulos ," Distance and similarity measures for fuzzy operators" , Inform.sci , 177(2007) 2336-2348.

بعض الخصائص للضرب الديكارتي لفضائيين معياريين ضبابيين

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الخلاصة

في هذا البحث مفهوم الضـــرب الـــديكارتي لفضـــائيين معياريين(قياســـيين) ضبابيين تم تقديمه. بعض النتائج (الخصائص) الاساسية والمبرهنـــات حـــول هـــذا المفهوم تم برهانها. الهدف الرئيسي لهذا البحث هو برهان أن الضرب الديكارتي لفضائيين معياريين ضبابيين تامين هو فضاء معياري ضبابي تام.